MATH 209 : Final Exam Solutions (Winter 2014)

1

(a) $\lim_{x\to 1^-} \frac{|x-1|}{x-1} =$

if x is close to 1 but smaller than 1 (less than), then $x-1$ is a smaller negative.
So, $|x-1|$ will be positive and $x-1$ will be "the same number" but negative.

$$\lim_{x\to 1^-} \frac{|x-1|}{x-1} = \boxed{-1}$$

(b) $$\lim_{x\to -2} \frac{(x+2)^2}{x^2-4} = \lim_{x\to -2} \frac{\cancel{(x+2)}(x+2)}{\cancel{(x+2)}(x-2)} = \lim_{x\to -2} \frac{x+2}{x-2} = \frac{(-2)+2}{(-2)-2} = \frac{0}{-4} = \boxed{0}$$

(c) $$\lim_{x\to\infty} \frac{x^2+4}{4-25x^2} = \lim_{x\to\infty} \frac{\frac{\cancel{x^2}}{\cancel{x^2}}+\frac{4}{x^2}}{\frac{4}{x^2}-\frac{25\cancel{x^2}}{\cancel{x^2}}} = \lim_{x\to\infty} \frac{1+4/x^2}{4/x^2-25}$$

$$= \frac{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \cancel{4/x^2}^{\,0}}{\lim_{x\to\infty} \cancel{4/x^2}^{\,0} - \lim_{x\to\infty} 25} = \boxed{\frac{1}{-25}}$$

2

(a) $y = 5x^{-7} - 2x^{-4}$

$$\frac{dy}{dx} = \frac{d}{dx}(5x^{-7}-2x^{-4}) = \frac{d}{dx}(5x^{-7}) - \frac{d}{dx}(2x^{-4})$$

$$= 5(-7)x^{-8} - 2(-4)x^{-5}$$

$$= \boxed{-35x^{-8} + 8x^{-5}}$$

OR

$$= \boxed{\frac{-35}{x^8} + \frac{8}{x^5}}$$

(b) $y = \frac{5}{x^{1/5}} - \frac{8}{x^{3/2}}$

$$\frac{d}{dx}y = \frac{d}{dx}\left(5x^{-1/5} - 8x^{-3/2}\right)$$

$$= \frac{d}{dx}\left(5x^{-1/5}\right) - \frac{d}{dx}\left(8x^{-3/2}\right)$$

$$= 5\left(-\frac{1}{5}\right)x^{(-1/5-1)} - 8\left(-3/2\right)x^{(-3/2-1)}$$

$$= \boxed{(-1)\,x^{-6/5} + 12\,x^{-5/2}}$$

$$= \boxed{\frac{-1}{x^{6/5}} + \frac{12}{x^{5/2}}}$$

OR

(c) $y = \frac{2x^5 - 4x^3 + 2x}{x^3} = \frac{2x^5}{x^3} - \frac{4x^3}{x^3} + \frac{2x}{x^3} = 2x^2 - 4 + 2x^{-2}$

$$\frac{d}{dx}y = \frac{d}{dx}\left(2x^2 - 4 + 2x^{-2}\right) = \frac{d}{dx}(2x^2) - \frac{d}{dx}(4) + \frac{d}{dx}(2x^{-2})$$

$$= 2(2)x - 0 + 2(-2)x^{-3} = \boxed{4x - 4x^{-3}} = \boxed{4x - \frac{4}{x^3}}$$

OR

(d) $y = (1+e^x)\ln x$

$$\frac{d}{dx}y = \frac{d}{dx}\left((1+e^x)\ln x\right) = \frac{d}{dx}(1+e^x)\cdot\ln x + (1+e^x)\cdot\frac{d}{dx}(\ln x)$$

$$= e^x\cdot\ln x + (1+e^x)\cdot\frac{1}{x} = \boxed{e^x\cdot\ln x + \frac{1+e^x}{x}}$$

(e) $y = \dfrac{\log_2 x}{1+x^2}$

$$\frac{d}{dx} y = \frac{d}{dx}\left(\frac{\log_2 x}{1+x^2}\right) = \frac{\frac{d}{dx}(\log_2 x)\cdot(1+x^2) - (\log_2 x)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{\frac{1}{x \cdot \ln 2}\cdot(1+x^2) + (\log_2 x)(0+2x)}{(1+x^2)^2}$$

$$= \boxed{\frac{1}{x(1+x^2)\ln 2} + \frac{2x \log_2 x}{(1+x^2)^2}}$$

(f) $y = 2\ln(x^2 - 3x + 4)$

$$\frac{d}{dx} y = \frac{d}{dx}\left(2\ln(x^2-3x+4)\right) = 2\frac{d}{dx}\left(\ln(x^2-3x+4)\right)$$

$$= 2\left[\frac{1}{x^2-3x+4}\cdot\frac{d}{dx}(x^2-3x+4)\right]$$

$$= \frac{2}{x^2-3x+4}(2x-3+0) = \boxed{\frac{4x-6}{x^2-3x+4}}$$

3

$xe^y - y = x^2 - 2 \quad \longrightarrow \quad$ Assume y is a function of x

$$\frac{d}{dx}\left(xe^{y(x)} - y(x)\right) = \frac{d}{dx}(x^2-2)$$

$$\frac{d}{dx}(xe^{y(x)}) - \frac{d}{dx}(y(x)) = \frac{d}{dx}(x^2) - \frac{d}{dx}(2)$$

$$e^y + xy' \cdot e^y - y' = 2x - 0$$

$$e^y + y'(x\cdot e^y - 1) = 2x$$

$$y'(xe^y - 1) = 2x - e^y$$

$$\boxed{y' = \frac{2x - e^y}{xe^y - 1}}$$

(4) $x = f(p) = 9500 - 250p$, $f'(p) = -250$

(a) Elasticity E

$x = f(p) = 9500 - 250p = 250(38 - p)$

[Note: since $x \geq 0$ and $p \geq 0$, $38 - p \geq 0$ $\Rightarrow$ $0 \leq p \leq 38$
$p \leq 38$]

let's find the elasticity of demand:

$$\boxed{E(p) = -\frac{p \cdot f'(p)}{f(p)}} = \frac{-p(-\cancel{250})}{\cancel{250}(38-p)} = \frac{+p}{38-p}$$

if $p = 15$, then $E(15) = \frac{-15}{38-(15)} = \frac{-15}{23} \approx -0.6521\ldots \not< -1$

since the elasticity of demand is between 0 & 1,
then the demand is inelastic.

(b) since the demand is inelastic,
a price increase will increase revenue
& a price decrease will decrease revenue.

5 $f(x) = x^4(x-6)^2$

(a) CRITICAL VALUES OF f (ie. when does $f'(x)=0$?)

$f'(x) = \frac{d}{dx} f(x) =$ use product Rule

$= \frac{d}{dx}(x^4)\cdot(x-6)^2 + (x^4)\cdot\frac{d}{dx}(x-6)^2$

$= 4x^3\cdot(x-6)^2 + x^4\cdot 2(x-6)\cdot(1)$

$= 4x^3(x-6)^2 + 2x^4(x-6)$

$= 2x^3(x-6)\left(2(x-6)+x\right)$

$= 2x^3(x-6)(3x-12) = 2x^3\cdot(x-6)\cdot 3(x-4) = 6x^3(x-6)(x-4)$

$f'(x)=0$?

$\underbrace{6x^3}_{\boxed{x=0}}\underbrace{(x-6)}_{\boxed{x=6}}\underbrace{(x-4)}_{\boxed{x=4}} = 0$

so, the critical values of f are $\boxed{\begin{array}{l} x=0 \\ x=4 \\ x=6 \end{array}}$

Intervals	$(-\infty, 0)$	$(0,4)$	$(4,6)$	$(6,\infty)$
(Test value in the interval) x	-1	1	5	7
$f'(x)$	$f'(-1) = -210$ (NEGATIVE)	$f'(1) = 90$ (POSITIVE)	$f'(5) = -750$ (NEGATIVE)	$f'(7) = 6174$ (POSITIVE)
$f(x)$	decreasing	increasing	decreasing	increasing

Note (When $f'(x)$ is positive, $f(x)$ is increasing. When $f'x)$ is negative, $f(x)$ is decreasing.)

(b) $\boxed{f(x) \text{ is increasing on: } (0,4)\cup(6,\infty)}$

(c) $\boxed{f(x) \text{ is decreasing on: } (-\infty,0)\cup(4,6)}$

(d) From increasing to decreasing : local maxima ⇒ at $x=4$
From decreasing to increasing : local minima ⇒ at $x=0$ and $x=6$

⑥ $g(x) = \ln(x^2 - 2x + 10)$

Recall: For the interval (a, b):
- $g(x)$ is concave upward if $g'(x)$ is increasing (ie. if $g''(x)$ is positive)
- $g(x)$ is concave downward if $g'(x)$ is decreasing (ie. if $g''(x)$ is negative)

$$g'(x) = \frac{d}{dx} g(x) = \frac{d}{dx}\left(\ln(x^2-2x+10)\right)$$

$$= \frac{1}{x^2-2x+10} \cdot \frac{d}{dx}(x^2-2x+10)$$

$$= \frac{1}{x^2-2x+10} \cdot (2x-2) \Rightarrow \boxed{g'(x) = \frac{2x-2}{x^2-2x+10}}$$

$$g''(x) = \frac{d}{dx} g'(x) = \frac{d}{dx}\left(\frac{2x-2}{x^2-2x+10}\right)$$

$$= \frac{\frac{d}{dx}(2x-2)\cdot(x^2-2x+10) - (2x-2)\cdot\frac{d}{dx}(x^2-2x+10)}{(x^2-2x+10)^2}$$

$$= \frac{2(x^2-2x+10) - (2x-2)(2x-2)}{(x^2-2x+10)^2}$$

OR

$$\boxed{g''(x) = \frac{2}{x^2-2x+10} - \frac{(2x-2)^2}{(x^2-2x+10)^2}}$$

(a) concave upward $\Rightarrow g''(x) > 0$

$\boxed{1^{st}}$: Find the critical value of $g'(x)$ (ie. where $g''(x) = 0$)

$$g''(x) = \frac{2(x^2-2x+10) - (2x-2)^2}{(x^2-2x+10)^2} = 0$$

if $2(x^2-2x+10) - (2x-2)^2 = 0$

$2x^2 - 4x + 20 - (4x^2 - 8x + 4) = 0$

$2x^2 - 4x + 20 - 4x^2 + 8x - 4 = 0$

$-2x^2 + 4x + 16 = 0$

$$-2x^2+4x+16=0$$

↳ use quadratic formula to find the zeros

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(4) \pm \sqrt{(4)^2-4(-2)(16)}}{2(-2)}$$

$$= \frac{-4 \pm \sqrt{144}}{-4} = \frac{-4 \pm 12}{-4} = \begin{cases} \frac{-4+12}{-4} = \boxed{-2} \\ \frac{-4-12}{-4} = \boxed{4} \end{cases}$$

So the critical values of $g'(x)$ are: $x = -2, 4$

(Note: The denominator $(x^2-2x+10)$ of $g'(x)$ is never zero for any value of x since its discriminant is negative.)

Interval	$(-\infty, -2)$	$(-2, 4)$	$(4, \infty)$
(Test value in interval) x	-3	0	6
$g''(x)$	$g''(-3) = -\frac{14}{625}$ (Negative)	$g''(0) = \frac{4}{25}$ (positive)	$g''(6) = -\frac{48}{289}$ (negative)
$g'(x)$	decreasing	increasing	decreasing
$g(x)$	concave downward	concave upward	concave downward.

(a) So, the interval where $g(x)$ is concave upward: $(-2, 4)$

(b) The interval where $g(x)$ is concave downward: $(-\infty, -2) \cup (4, \infty)$

(c) Recall:
Infection point(s): a point(s) on the graph of a function where the concavity changes

according to the table on (a), infection points are: $\boxed{x=-2}$ $\boxed{x=4}$

⑦ $f(x) = x^4 - 8x^2 + 16$ on $[-3, 4]$

→ Find the critical value of $f(x)$ (ie. $f'(x) = 0$)

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx}\left(x^4 - 8x^2 + 16\right) = 4x^3 - 16x = 4x\left(x^2 - 4\right) = 4x(x+2)(x-2)$$

when $f'(x) = 0$?

$$f'(x) = \underbrace{4x}_{\boxed{x=0}}\underbrace{(x+2)}_{\boxed{x=-2}}\underbrace{(x-2)}_{\boxed{x=2}} = 0$$

] all in the interval $[-3, 4]$

Interval	$(-3, -2)$	$(-2, 0)$	$(0, 2)$	$(2, 4)$
(Test value in interval) x	-2.5	-1	1	3
$f'(x)$	$f'(-2.5) = -22.5$ (negative)	$f'(-1) = 12$ (positive)	$f'(1) = -12$ (negative)	$f'(3) = 60$ (positive)
$f(x)$	decreasing	increasing	decreasing	increasing

min point at $x = -2$ max point at $x = 0$ min point at $x = 2$

Recall, absolute extrema are the largest and smallest the function will ever be.

Look at the critical points & the endpoints of the given interval:

$f(-2) = 0$ $f(-3) = 25$

$f(0) = 16$ $f(4) = 144$

$f(2) = 0$

In this case, absolute maxima of $f(x)$ is 144 at $x = 4$ (endpoint)
absolute minima of $f(x)$ is 0 at $x = -2$ & $x = 2$
(critical points)

(8)

(a) $\int_{-5}^{5} (10 - 7x + x^2)\,dx = 10x - \frac{7}{2}x^2 + \frac{1}{3}x^3 \Big|_{-5}^{5}$

$= \left(10(5) - \frac{7}{2}(5)^2 + \frac{1}{3}(5)^3\right) - \left(10(-5) - \frac{7}{2}(-5)^2 + \frac{1}{3}(-5)^3\right)$

$= \left(50 - \cancel{\frac{175}{2}} + \frac{125}{3}\right) - \left(-50 - \cancel{\frac{175}{2}} - \frac{125}{3}\right)$

$= 100 + \frac{250}{3} = \frac{300}{3} + \frac{250}{3} = \frac{550}{3} \approx \boxed{183.333}$

(b) $\int_0^1 x e^{-2x^2}\,dx$

let $t = -2x^2 \Rightarrow e^{-2x^2} = e^t$

$dt = -4x\,dx \Rightarrow x\,dx = \frac{-1}{4}dt$

$\int x e^{-2x^2} dx = \int \frac{-1}{4} e^t dt = \frac{-1}{4}\int e^t dt = \frac{-1}{4} e^t$

since $t = -2x^2$, $-\frac{1}{4}e^t = -\frac{1}{4}e^{-2x^2}$

So $\int_0^1 x e^{-2x^2} dx = \left(-\frac{1}{4}e^{-2x^2}\right)\Big|_0^1 = \left(-\frac{1}{4}e^{-2(1)^2}\right) - \left(-\frac{1}{4}e^{-2(0)^2}\right)$

$= -\frac{1}{4}e^{-2} + \frac{1}{4}e^0$

$= -\frac{1}{4}e^{-2} + \frac{1}{4} \approx \boxed{0.216}$

(c) $\int_0^3 \frac{x}{(1+x^2)^2}\,dx$

let $t = 1+x^2 \Rightarrow \frac{1}{(1+x^2)^2} = \frac{1}{t^2}$

$dt = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}dt$

[Note: for $x=3$, $t=10$; $x=0$, $t=1$]

so, $\int_0^3 \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2}\int_1^{10} \frac{1}{t^2}dt = \frac{1}{2}\int_1^{10} t^{-2}dt = \frac{1}{2}\left(-t^{-1}\right)\Big|_1^{10} = -\frac{1}{2t}\Big|_1^{10}$

$= \left(\frac{-1}{2(10)}\right) - \left(\frac{-1}{2(1)}\right) = \frac{-1}{20} + \frac{1}{2} =$

$= \frac{-1}{20} + \frac{10}{20} = \boxed{\frac{9}{20}}$

$\approx \boxed{0.450}$

9

(a) $\int \frac{x^2e^x - 2x}{x^2}\,dx$

$= \int \frac{\cancel{x^2}e^x}{\cancel{x^2}}dx - \int \frac{2x}{x^2}dx = \int e^x dx - 2\int \frac{1}{x}dx = \boxed{e^x - 2\ln x + C}$

(b) $\int \frac{x}{\sqrt{x+5}}\,dx$ let $t = x+5 \Rightarrow x = t-5$

$dt = 1 \cdot dx$

$\int \frac{t-5}{t^{1/2}}dt = \int t^{1/2}dt - 5\int t^{-1/2} = \frac{2}{3}t^{3/2} - 5(2t^{1/2})$

$= \frac{2}{3}t^{3/2} - 10t^{1/2} = 2t^{1/2}\left(\frac{1}{3}t - 5\right)$

Since $t = x+5$, $2t^{1/2}\left(\frac{1}{3}t - 5\right) = 2\sqrt{x+5}\left(\frac{1}{3}(x+5) - 5\right)$

$= 2\sqrt{x+5}\left(\frac{1}{3}x - \frac{10}{3}\right)$

so, $\boxed{\int \frac{x}{\sqrt{x+5}}dx = \frac{2}{3}\sqrt{x+5}\,(x-10) + C}$

(c) $\int x^3(2x^4+5)^5\,dx$

let $t = 2x^4+5$
$dt = 8x^3\,dx$
$\Rightarrow x^3\,dx = \frac{1}{8}dt$

so : $\int x^3(2x^4+5)^5\,dx =$

$= \frac{1}{8}\int t^5\,dt = \frac{1}{8}\left(\frac{1}{6}t^6\right) + C$

$= \frac{1}{48}t^6 + C$

since $t = 2x^4+5$,

$\frac{1}{48}t^6 = \frac{1}{48}(2x^4+5)^6$

Thus $\boxed{\int x^3(2x^4+5)^5\,dx = \frac{1}{48}(2x^4+5)^6 + C}$

(d) $\int \frac{e^{-x}}{(e^{-x}+3)}\,dx$

let $t = e^{-x}+3$
$dt = -e^{-x}\,dx$
$\Rightarrow e^{-x}\,dx = -1\cdot dt$

so $\int \frac{e^{-x}}{(e^{-x}+3)}\,dx = \int \frac{-1}{t}\,dt$

$= -\int \frac{1}{t}\,dx = -\ln(t) + C$

since $t = e^{-x}+3$, $-\ln(e^{-x}+3)$

Thus, $\boxed{\int \frac{e^{-x}}{(e^{-x}+3)}\,dx = -\ln(e^{-x}+3) + C}$

(10) Area bounded by $y = x^3+1$ & $y = x+1$

1^{st}: <u>when does $f(x) = g(x)$??</u>

let $f(x) = x^3+1$

$g(x) = x+1$

$$x^3+\cancel{1} = x+\cancel{1}$$

$$x^3 - x = 0$$

$$x(x^2-1) = 0$$

$$\underbrace{x}_{\boxed{x=0}}\underbrace{(x+1)}_{\boxed{x=-1}}\underbrace{(x-1)}_{\boxed{x=1}} = 0$$

The area we want is in the interval $[-1, 1]$

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
(Test value in interval) x	-2	-0.5	0.5	2
$f(x)$	$f(-2) = -7$	$f(-0.5) = 0.875$	$f(0.5) = 1.125$	$f(2) = 9$
$g(x)$	$g(-2) = -1$	$g(-0.5) = 0.5$	$g(0.5) = 1.5$	$g(2) = 3$
compare $f(x)$ & $g(x)$	$f(x) < g(x)$	$g(x) < f(x)$	$f(x) < g(x)$	$g(x) < f(x)$

so, on $(-\infty,-1) \cup (0,1)$, $f(x)$ is on top of $g(x)$ (ie. $f(x) > g(x)$)

$(-1,0) \cup (1,\infty)$, $g(x)$ is on top of $f(x)$ (ie. $g(x) > f(x)$)

Then,

$$\text{Total Area} = \int_{-1}^{0} (f(x) - g(x))\,dx + \int_{0}^{1} (g(x) - f(x))\,dx$$

$$= \int_{-1}^{0} [(x^3+1) - (x+1)]\,dx + \int_{0}^{1} [(x+1) - (x^3+1)]\,dx$$

$$= \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx$$

$$= \left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Big|_{-1}^{0} + \left(\frac{1}{2}x^2 - \frac{1}{4}x^4\right)\Big|_{0}^{1}$$

$$= 0 - \left(\frac{1}{4}(-1)^4 - \frac{1}{2}(-1)^2\right) + \left(\frac{1}{2}(1)^2 - \frac{1}{4}(1)^4\right) - 0$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \boxed{\frac{1}{2}}$$

(11)

$A = P \cdot e^{rt}$

$r = ?$

$t = 100$ years

$A = 2P$ ~ population doubles

$$2P = P \cdot e^{100r}$$

$$2 = e^{100r}$$

$$\ln(2) = \ln(e^{100r})$$

$$\ln(2) = 100 \cdot r$$

$$\frac{\ln(2)}{100} = r \rightarrow r \approx 0.006931$$

so rate of $\boxed{0.69\%}$