MAT1332, Winter 2011, Assignment 1 Solutions

Total=7 points.

1. **(1 point)** (a) Use the substitution $u = 3 - 14t$. Then $\frac{du}{dt} = -14$, so $dt = \frac{du}{-14}$. Thus

$$\int \frac{1}{3-14t}dt = -\frac{1}{14}\int \frac{1}{u}du = -\frac{1}{14}\ln|u| + C = -\frac{1}{14}\ln|3-14t| + C$$

(Don't forget to resubstitute - and don't forget the absolute value signs in the logarithm or the $+C$!)

(b)

$$\int_{-3}^{3}(y^7 - 2y^9)dy = \left[\frac{y^8}{8} - \frac{y^{10}}{5}\right]_{-3}^{3} = \left[\frac{3^8}{8} - \frac{3^{10}}{10}\right] - \left[\frac{(-3)^8}{8} - \frac{(-3)^{10}}{10}\right] = 0\,.$$

2. (a)

$$\int_{-\pi}^{\pi}[x^2 - 30\cos(x)]dx = \left[\frac{x^3}{3} - 30\sin x\right]_{-\pi}^{\pi} = \left[\frac{\pi^3}{3} - 0\right] - \left[\frac{(-\pi)^3}{3} - 0\right] = \frac{2\pi^3}{3}\,.$$

(b) **(1 point)** *First approach* First find the indefinite integral by using the substitution $u = 3\pi(x-5)$. Then $\frac{du}{dx} = 3\pi$, so $dx = \frac{du}{3\pi}$. Hence

$$3\int \sin(3\pi(x-5)dx = 3\int \sin u\frac{du}{3\pi} = -\frac{1}{\pi}\cos u + C = -\frac{1}{\pi}\cos(3\pi(x-5)) + C\,.$$

Then evaluate

$$3\int_{2}^{5}\sin(3\pi(x-5)dx = -\frac{1}{\pi}\cos(3\pi(x-5))\Big|_{2}^{5} = -\frac{1}{\pi}[\cos(0) - \cos(-9\pi)] = -\frac{1}{\pi}[1+1] = -\frac{2}{\pi}\,.$$

Second approach Transform the limits of integration first. When $x = 2$, $u = 3\pi(2-5) = -9\pi$. When $x = 5$, $u = 3\pi(5-5) = 0$. Then the integral after substitution becomes

$$3\int_{2}^{5}\sin(3\pi(x-5)dx = 3\int_{-9\pi}^{0}\sin u\frac{du}{3\pi} = -\frac{1}{\pi}[\cos(0) - \cos(-9\pi)] = -\frac{2}{\pi}\,.$$

3. **(1 point)** Use the substitution $u = \arctan x$. Then $\frac{du}{dx} = \frac{1}{1+x^2}$ so $dx = (1+x^2)du$. Hence

$$\int_0^1 \frac{e^{\arctan x}}{1+x^2}dx = \int_{x=0}^{x=1}\frac{e^u}{1+x^2}(1+x^2)du = \int_{x=0}^{x=1}e^u du = e^u\Big|_{x=0}^{x=1} = e^{\arctan x}\Big|_0^1 = e^{\pi/4} - 1 = 0.284025\,.$$

4. **(1 point)** Using integration by parts, we have

$$u = x \qquad v' = \sin 3x$$

$$u' = 1 \qquad v = -\frac{\cos 3x}{3}$$

Thus

$$\begin{aligned}
\int_0^\pi x\sin(3x)dx &= -\frac{1}{3}x\cos 3x\Big|_0^\pi + \frac{1}{3}\int_0^\pi \cos 3x dx \\
&= \left[-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x\right]_0^\pi \\
&= \left[-\frac{1}{3}\pi\cos 3\pi + \frac{1}{9}\sin 3\pi\right] - \left[-\frac{1}{3}(0) - \frac{1}{9}\sin 0\right] \\
&= \frac{\pi}{3}
\end{aligned}$$

5. To answer the questions, we need to solve the zombie equation. Since we'll need to do this for a variety of scenarios, it's best to find the indefinite integral first. We'll need to use integration by parts, so we have

$$u = 10t \qquad v' = e^{-0.08t}$$

$$u' = 10 \qquad v = -\frac{e^{-0.08t}}{0.08}$$

Thus

$$\begin{aligned}
z &= -\frac{10te^{-0.08t}}{0.08} + \frac{10}{0.08}\int e^{-0.08t}dt \\
&= -\frac{10te^{-0.08t}}{0.08} - \frac{10}{0.08^2}e^{-0.08t} + C
\end{aligned}$$

Don't forget the $+C$, it's absolutely crucial. Forget the $+C$ and you'll be devoured by a zombie.

What happens now? We almost have the solution, but not quite (we don't know what C is). And we haven't used the initial condition $z(0) = 5$. Let's put that in and see what happens:

$$z(0) = -0 - \frac{10}{0.08^2}e^0 + C = -\frac{10}{0.08^2} + C = 5\,.$$

Thus,

$$C = 5 + \frac{10}{0.08^2}\,.$$

and hence

$$z(t) = -\frac{10te^{-0.08t}}{0.08} - \frac{10}{0.08^2}e^{-0.08t} + 5 + \frac{10}{0.08^2}\,.$$

This is the solution because it tells us how many zombies exist at any given time. There's only z and t to be determined; everything else is known. We're now in a position to answer our sub-questions.

(a) During the first week, there will be $z(7) - z(0)$ zombies recruited.

$$z(7) - z(0) = -\frac{10(7)e^{-0.08(7)}}{0.08} - \frac{10}{0.08^2}e^{-0.08(7)} + 5 + \frac{10}{0.08^2} - 5 = 170.17$$

Thus, there are 170 zombies recruited during the first week. They may be undead, but the zombies are very efficient.
(It's not 175, since the original 5 weren't recruited. That's why we have to subtract $z(0)$.)

(b) **(1 point)** During the third week there are $z(21)$ zombies in total, but $z(14)$ of them already existed at the end of the second week. Thus, during the third week, $z(21) - z(14)$ zombies are recruited.

$$\begin{aligned} z(21) - z(14) &= \left[-\frac{10(21)e^{-0.08(21)}}{0.08} - \frac{10}{0.08^2} e^{-0.08(21)} + 5 + \frac{10}{0.08^2} \right] \\ &\quad - \left[-\frac{10(14)e^{-0.08(14)}}{0.08} - \frac{10}{0.08^2} e^{-0.08(14)} + 5 + \frac{10}{0.08^2} \right] \\ &= 300.3608 \,. \end{aligned}$$

Thus, there are 300 zombies recruited in the third week. This is a lot; it's getting dangerous on campus!

(And you can see the fundamental theorem of calculus at work here. Now which is scarier: math or zombies?)

(c) After 50 days, the total number of zombies is

$$z(50) = -\frac{10(50)e^{-0.08(50)}}{0.08} - \frac{10}{0.08^2} e^{-0.08(50)} + 5 + \frac{10}{0.08^2} = 1424.4091 \,.$$

Fifty days into the semester, if you visit campus, you'll see a lot of pale, shambling figures, their eyes glazed over and making moaning sounds. You'll also see 1424 zombies.

(d) **(2 points)**

After a long time, like 1000 days, we have

$$\begin{aligned} z(1000) - z(0) &= -\frac{10(1000)e^{-0.08(1000)}}{0.08} - \frac{10}{0.08^2} e^{-0.08(1000)} + 5 + \frac{10}{0.08^2} - 5 \\ &= 1562.5 - 2.28 \times 10^{-30} \\ &\approx 1562.5 \end{aligned}$$

(Don't forget to subtract the original five zombies, also known as $z(0)$!)

So now we know: the initial five zombies will wreak some devastation, infecting 1562 humans, but they won't take over the whole campus.

(Be careful here: the answer isn't 1563 because they never quite finish that last half person, so she doesn't count as a zombie.)

Calculus can help you avoid the impending zombie plague. Who said math was useless?