Solutions for sample midterm - Fall 2016

Question 1

Part a

Write the sigma notation formula for the right Riemann sum R_n of the function $f(x) = 4 - x^2$ on the interval $[0, 2]$ using n sub-intervals of equal length, and calculate the definite integral $\int_0^2$ as the limit of R_n at $n \to \infty$.

(Reminder $\sum_{k=1}^{n} k = n(n+1)/2, \quad \sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$

Solution

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i = a + i\Delta x = 0 + i\frac{2}{n} = \frac{2i}{n}$$

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \sum_{i=1}^{n} \left[4 - \left(\frac{2i}{n}\right)^2\right] \frac{2}{n} \\ &= \frac{2}{n}\left[\sum_{i=1}^{n} 4 - \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2\right] \\ &= \frac{2}{n}\left[4n - \sum_{i=1}^{n} \frac{4}{n^2} i^2\right] \\ &= 8 - \frac{8}{n^3}\left[\sum_{i=1}^{n} i^2\right] \\ &= 8 - \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\ &= 8 - \frac{4n(n+1)(2n+1)}{3n^3} \end{aligned}$$

$$\int_0^2 4 - x^2 \, dx = \lim_{n\to\infty} \left[8 - \frac{4n(n+1)(2n+1)}{3n^3}\right]$$

$$= 8 - \frac{(4)(2)}{3}$$
$$= \frac{16}{3}$$

Part b

Use the Fundamental Theorem of Calculus to calculate the derivative of

$$F(x) = \int_{e^{-x}}^{x} \ln(t^2+1)\ dt$$

Solution

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

$$\int_{e^{-x}}^{x} \ln(t^2+1)\ dt = \int_0^x \ln(t^2+1)\ dt - \int_0^{e^{-x}} \ln(t^2+1)\ dt$$

By using the chain rule

$$F'(x) = \frac{dF}{dx} = \frac{dF_1}{du}\frac{du}{dx} + \frac{dF_2}{dy}\frac{dy}{dx}$$

$$u = x \qquad\qquad y = e^{-x}$$
$$\frac{du}{dx} = 1 \qquad\qquad \frac{dy}{dx} = -e^{-x}$$

$$F'(x) = \frac{d}{dx}\int_0^x \ln(t^2+1)\ dt \quad - \frac{d}{dx}\int_0^{e^{-x}} \ln(t^2+1)\ dt$$
$$= \frac{d}{du}\int_0^u \ln(t^2+1)\ dt\ \frac{du}{dx} - \frac{d}{dy}\int_0^y \ln(t^2+1)\ dt\ \frac{dy}{dx}$$

$$= \ln(u^2+1) - \ln(y^2+1)\left(-e^{-x}\right)$$

$$= \ln(x^2+1) + \ln(e^{-2x}+1)\left(e^{-x}\right)$$

Question 2

Calculate the following indefinite integrals

Part a

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx$$

Solution

$$u = x^2, \quad du = 2x\,dx$$

$$\begin{aligned}\int \frac{x^3}{\sqrt{16-x^2}}\,dx &= \int \frac{ux}{\sqrt{16-u}}\,\frac{du}{2x} \\ &= \frac{1}{2}\int \frac{u}{\sqrt{16-u}}\,du\end{aligned}$$

$$y = 16-u, \quad dy = -du, \quad u = 16-y$$

$$\begin{aligned}\int \frac{x^3}{\sqrt{16-x^2}}\,dx &= -\frac{1}{2}\int \frac{16-y}{\sqrt{y}}\,dy \\ &= -\frac{1}{2}\int 16y^{-0.5} - y^{0.5}\,dy \\ &= -\frac{1}{2}\frac{16y^{0.5}}{0.5} - \frac{y^{1.5}}{1.5} + C \\ &= -\frac{1}{2}\frac{16y^{0.5}}{0.5} + \frac{1}{2}\frac{y^{1.5}}{1.5} + C \\ &= -16y^{0.5} + \frac{y^{1.5}}{3} + C \\ &= -16(16-u)^{0.5} + \frac{(16-u)^{1.5}}{3} + C \\ &= -16\sqrt{16-x^2} + \frac{(\sqrt{16-x^2})^3}{3} + C\end{aligned}$$

Another solution

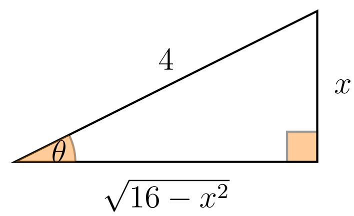
$$
\begin{aligned}
x &= 4\sin\theta \\
dx &= 4\cos\theta\, d\theta
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{16-x^2} &= \sqrt{16-(4\sin\theta)^2} \\
&= \sqrt{16-16\sin^2\theta} \\
&= \sqrt{16(1-\sin^2\theta)} \\
&= \sqrt{16\cos^2\theta} \\
&= 4\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{x^3}{\sqrt{16-x^2}}\,dx &= \int \frac{(4\sin\theta)^3}{4\cos\theta}\,4\cos\theta\,d\theta \\
&= \int (4\sin\theta)^3\,d\theta \\
&= 64\int \sin^3\theta\,d\theta \\
&= 64\int \sin\theta\sin^2\theta\,d\theta \\
&= 64\int \sin\theta(1-\cos^2\theta)\,d\theta
\end{aligned}
$$

$$
u = \cos\theta, \quad du = -\sin\theta\ d\theta
$$

$$
\begin{aligned}
\int \frac{x^3}{\sqrt{16-x^2}}\,dx &= 64\int \sin\theta(1-u^2)\,\frac{du}{-\sin\theta} \\
&= -64\int (1-u^2)\,du \\
&= -64\left(u-\frac{u^3}{3}\right)+C \\
&= -64\left(\cos\theta-\frac{(\cos\theta)^3}{3}\right)+C
\end{aligned}
$$



$$\begin{aligned}
\int \frac{x^3}{\sqrt{16-x^2}}\,dx &= -64\left(\frac{\sqrt{16-x^2}}{4} - \frac{(\frac{\sqrt{16-x^2}}{4})^3}{3}\right) + C \\
&= -64\left(\frac{\sqrt{16-x^2}}{4} - \frac{(\sqrt{16-x^2})^3}{192}\right) + C \\
&= -16\sqrt{16-x^2} + \frac{(\sqrt{16-x^2})^3}{3} + C
\end{aligned}$$

Part b

$$\int \frac{x^2+3}{x^2-2x+5}\,dx$$

Solution

This looks like a partial fractions problem, but the polynomial in the numerator is of order 2, which is the same as the order of the polynomial in the denominator, so we start by long division.

$$\begin{array}{r|r}
 & 1 \\ \hline
x^2-2x+5 & x^2 \qquad +3 \\
 & -x^2+2x-5 \\ \hline
 & 2x-2
\end{array}$$

Therefore, the original integral can be written as follows:

$$\int \frac{x^2+3}{x^2-2x+5}\,dx = \int 1 + \frac{2x-2}{x^2-2x+5}\ dx$$

$$\begin{aligned}
u &= x^2-2x+5 \\
du &= 2x-2\ dx
\end{aligned}$$

$$\begin{aligned}\int \frac{x^2+3}{x^2-2x+5}\,dx &= \int 1\ dx + \int \frac{1}{u}\ du\\ &= x + \ln(u) + C\\ &= x + \ln(x^2-2x+5) + C\end{aligned}$$

Part c

$$\int \frac{2^x}{2^{2x}-4}\,dx$$

Solution

$$\begin{aligned}u &= 2^x\\ du &= 2^x \ln(2)\ dx\end{aligned}$$

$$\begin{aligned}\int \frac{2^x}{2^{2x}-4}\,dx &= \int \frac{1}{u^2-4}\,\frac{du}{ln(2)}\\ &= \frac{1}{ln(2)}\int \frac{1}{u^2-4}\,du\end{aligned}$$

This looks like a partial fractions problem now, so we need to get the factors of the denominator

$$u^2-4=(u-2)(u+2)$$

Therefore, the fraction can be written as follows:

$$\begin{aligned}\frac{1}{(u+2)(u-2)} &= \frac{A}{(u-2)} + \frac{B}{(u+2)}\\ \text{if } u=2 \rightarrow\ 1 &= 4A\ \rightarrow\ A=\frac{1}{4}\\ \text{if } u=-2 \rightarrow\ 1 &= -4b \rightarrow\ B=-\frac{1}{4}\end{aligned}$$

Therefore, the integral will be written as follows:

$$\int \frac{2^x}{2^{2x}-4}\,dx = \frac{1}{ln(2)}\left[\int \frac{1}{4}\frac{1}{u-2}\,du - \int \frac{1}{4}\frac{1}{u+2}\,du\right]$$

$$
\begin{aligned}
&= \frac{1}{ln(2)}\left[\frac{1}{4}\ln|u-2| - \frac{1}{4}\ln|u+2|\right] + C \\
&= \frac{1}{4ln(2)}\ln\left|\frac{u-2}{u+2}\right| + C \\
&= \frac{1}{ln(16)}\ln\left|\frac{2^x-2}{2^x+2}\right| + C
\end{aligned}
$$

Question 3

Given that $F(0) = 0$, find the antiderivative

$$F(X) = \frac{\sec^2 x}{\sec^2(x) + 3}$$

Solution

$$
\begin{aligned}
\int \frac{\sec^2 x}{\sec^2(x)+3}\,dx &= \int \frac{\sec^2 x}{(\tan^2(x)+1)+3}\,dx \\
&= \int \frac{\sec^2 x}{\tan^2(x)+4}\,dx
\end{aligned}
$$

$$
\begin{aligned}
u &= \tan(x) \\
du &= \sec^2(x)dx
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{\sec^2 x}{\sec^2(x)+3}\,dx &= \int \frac{1}{u^2+4}\,du \\
&= \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C \\
&= \frac{1}{2}\arctan\left(\frac{\tan(x)}{2}\right) + C
\end{aligned}
$$

Question 4

Evaluate the following definite integrals (give the exact values, do not approximate):

Part a

$$\int_0^4 \frac{x^2}{\sqrt{2x+1}}\ dx$$

Solution

$$u = 2x+1 \qquad du = 2\ dx \qquad x = \left(\frac{u-1}{2}\right)$$

$$\begin{aligned}
\int_0^4 \frac{x^2}{\sqrt{2x+1}}\ dx &= \int_1^9 \frac{\left(\frac{u-1}{2}\right)^2}{\sqrt{u}}\ \frac{du}{2} \\
&= \frac{1}{8}\int_1^9 \frac{u^2-2u+1}{\sqrt{u}}\ du \\
&= \frac{1}{8}\int_1^9 u^{1.5} - 2u^{0.5} + u^{-0.5}\ du \\
&= \frac{1}{8}\left(\frac{u^{2.5}}{2.5} - 2\frac{u^{1.5}}{1.5} + \frac{u^{0.5}}{0.5}\right)\Bigg|_1^9 \\
&= \left(\frac{1}{20}u^{2.5} - \frac{1}{6}u^{1.5} + \frac{1}{4}u^{0.5}\right)\Bigg|_1^9 \\
&= \left(\frac{1}{20}9^{2.5} - \frac{1}{6}9^{1.5} + \frac{1}{4}9^{0.5}\right) - \left(\frac{1}{20} - \frac{1}{6} + \frac{1}{4}\right) \\
&= 8.4 - \frac{2}{15} \\
&= \frac{124}{15}
\end{aligned}$$

Part b

$$\int_0^1 x^2\cos(\pi x)\ dx$$

Solution

$$\begin{aligned}
u &= x^2 & dv &= \cos(\pi x)dx \\
du &= 2x\ dx & v &= \frac{\sin(\pi x)}{\pi}
\end{aligned}$$

$$\int_0^1 x^2\cos(\pi x)\ dx = x^2\ \frac{\sin(\pi x)}{\pi}\bigg|_0^1 - \frac{2}{\pi}\int_0^1 x\sin(\pi x)\ dx$$

$$\begin{aligned} u &= x & dv &= \sin(\pi x)dx \\ du &= 1\ dx & v &= \frac{-\cos(\pi x)}{\pi} \end{aligned}$$

$$\begin{aligned} \int_0^1 x^2\cos(\pi x)\ dx &= x^2\ \frac{\sin(\pi x)}{\pi}\bigg|_0^1 - \frac{2}{\pi}\left(\frac{-1}{\pi}x\ cos(\pi x)\bigg|_0^1 - \frac{-1}{\pi}\int_0^1 \cos(\pi x)\right) \\ &= x^2\ \frac{\sin(\pi x)}{\pi}\bigg|_0^1 + \frac{2}{\pi^2}\ x\ cos(\pi x)\bigg|_0^1 - \frac{2}{\pi^2}\int_0^1 \cos(\pi x) \\ &= x^2\ \frac{\sin(\pi x)}{\pi}\bigg|_0^1 + \frac{2}{\pi^2}\ x\ cos(\pi x)\bigg|_0^1 - \frac{2}{\pi^3}\sin(\pi x)\bigg|_0^1 \\ &= \left(0 + \frac{-2}{\pi^2} + 0\right) - (0+0+0) \\ &= \frac{-2}{\pi^2} \end{aligned}$$

Question 5

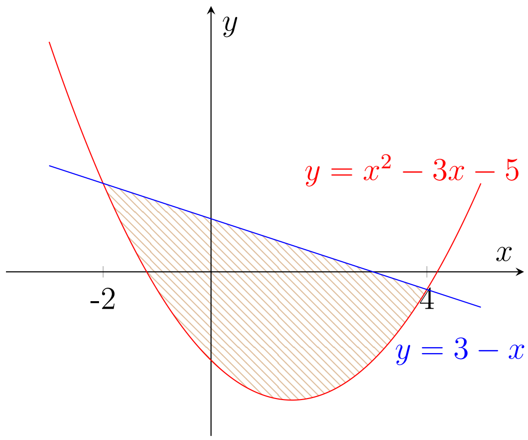
Find the area of the region enclosed by the curves $y = x^2 - 3x - 5$ and $y = 3 - x$.

Solution

First we find the points of intersection

$$\begin{aligned} x^2 - 3x - 5 &= 3 - x \\ x^2 - 3x - 5 - 3 + x &= 0 \\ x^2 - 2x - 8 &= 0 \\ (x-4)(x+2) &= 0 \\ x = 4 &\text{ and } x = -2 \end{aligned}$$

Now we graph the given functions.



$$
\begin{aligned}
A &= \int_{-2}^{4} (3-x) - (x^2 - 3x - 5)\, dx \\
&= \int_{-1}^{0} 3 - x - x^2 + 3x + 5\, dx \\
&= \int_{-1}^{0} -x^2 + 2x + 8\, dx \\
&= \frac{-1}{3}x^3 + \frac{2}{2}x^2 + 8x \Big|_{-2}^{4} \\
&= \left(\frac{-1}{3}4^3 + 4^2 + 8(4)\right) - \left(\frac{-1}{3}(-2)^3 + (-2)^2 + 8(-2)\right) \\
&= \left(\frac{-1}{3}(4)^3 + (4)^2 + 8(4)\right) - \left(\frac{-1}{3}(-2)^3 + (-2)^2 + 8(-2)\right) \\
&= \frac{80}{3} - \frac{-28}{3} \\
&= 36
\end{aligned}
$$

Bonus Question

Given that

$$\int_0^{\pi} [f(x) + f''(x)] \sin x \, dx = 2$$

and $f(\pi) = 1$, find $f(0)$.

Solution

$$\int_0^{\pi} [f(x) + f''(x)] \sin x \, dx = \int_0^{\pi} f(x) \sin x \, dx + \int_0^{\pi} f''(x) \sin x \, dx$$

$$u = \sin x \qquad dv = f''(x)\,dx$$
$$du = \cos x \qquad v = f'(x)$$

$$\int_0^\pi f(x)\sin x\,dx + \int_0^\pi f''(x)\sin x\,dx = \int_0^\pi f(x)\sin x\,dx + f'(x)\sin x\Big|_0^\pi - \int_0^\pi f'(x)\cos x\,dx$$

$$u = \cos x \qquad dv = f'(x)\,dx$$
$$du = -\sin x \qquad v = f(x)$$

$$\int_0^\pi f(x)\sin x\,dx + \int_0^\pi f''(x)\sin x\,dx = \int_0^\pi f(x)\sin x\,dx + f'(x)\sin x\Big|_0^\pi - \left(f(x)\cos x\Big|_0^\pi + \int_0^\pi f(x)\sin x\,dx \right)$$

$$\int_0^\pi f(x)\sin x\,dx + \int_0^\pi f''(x)\sin x\,dx = \int_0^\pi f(x)\sin x\,dx + f'(x)\sin x\Big|_0^\pi - f(x)\cos x\Big|_0^\pi - \int_0^\pi f(x)\sin x\,dx$$

$$\int_0^\pi f(x)\sin x\,dx + \int_0^\pi f''(x)\sin x\,dx = f'(x)\sin x\Big|_0^\pi - f(x)\cos x\Big|_0^\pi = 2$$
$$= (f'(\pi)\sin(\pi) - f'(0)\sin(0))$$
$$- (f(\pi)\cos(\pi) - f(0)\cos(0))$$

$$= f(\pi) + f(0) = 2$$
$$= 1 + f(0) = 2$$

$$f(0) = 1$$