CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number		Section(s)
Mathematics	208/4		All
Examination	**Date**	**Time**	**Pages**
Midterm	March 2018	1 Hour 30 minutes	2
Instructors			**Course Examiner**
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FORMULAE:

$$A = P(1+i)^n, \quad A = Pe^{rt}, \quad FV = PMT\,\frac{(1+i)^n - 1}{i}, \quad PV = PMT\,\frac{1-(1+i)^{-n}}{i}$$

Special Instructions:

- ▷ Answer all questions.
- ▷ **Only approved calculators are allowed.**

MARKS

R(X) = p(X) · X

[3+4+3] **1.** Given the following price-demand and cost functions:

$$p(x) = 27 - 3x, \qquad C(x) = 15x + 9,$$

where $p(x)$ is the wholesale price in dollars at which x thousand units of product can be sold and $C(x)$ is in thousands of dollars, both functions have the domain $0 \le x \le 9$,

(A) Form the revenue function $R(x)$ and find the break-even points;

(B) Form the profit function $P(x)$ and find the maximum profit;

(C) What price will maximize the profit?

R(x) = C(x)

[2.5 × 4] **2.** Solve for x in the following equations:

(A) $81(27^{x-2})^{x+1} = 3^{4x+4}$

(B) $\log_3(5+2x) - 1 = \frac{1}{2}\log_3 25$

(C) $2018^{7x^2-68x+138} - 2018^{2(x^2-8)+x} = 0$

(D) $\log_3(2x+3) + \log_3(x-1) = \log_3(4x-3)$

PLEASE TURN OVER

80% · 10% = 0.8 · 0.1 =
= 0.08 = 8%

[5+5] **3.** (A) If the 13^{th} and 38^{th} terms of an arithmetic sequence are -53 and -128, respectively, find the 668^{th} term of the sequence.

(B) The sum and the first term of an infinite geometric series are 6 and 5, respectively, find the 3^{rd} term of the series.

[5+5] **4.** Jeremy makes his first \$2,500 deposit into an IRA earning 6.4% compounded annually on his 23^{rd} birthday and his last \$2,500 deposit on his 48^{th} birthday (26 equal deposits in all). With no additional deposits, the money in the IRA continues to earn 6.4% interest compounded annually until Jeremy retires on his 64^{th} birthday.

(A) How much is in the IRA when Jeremy retires?

(B) How much interest is earned during the last six years before retirement?

[5+5] **5.** A family has a \$151,900, 20-year mortgage at 8.7% compounded monthly.

(A) Find the amount of the monthly payment and the total interest paid.

(B) Suppose the family decides to add an extra \$100 to its mortgage payment each month starting with the very first payment. How many years will it take to pay off the mortgage (round the answer to the nearest integer)? How much interest will the family save?

[3+3+4] **6.** Suppose that the supply and demand for printed baseball caps for a particular week are

$$p = 0.4q + 3.2 \quad \text{(Price-supply equation)}$$

$$p = -1.9q + 17 \quad \text{(Price-demand equation)}$$

where p is the price in dollars and q is the quantity in hundreds.

(A) Find the supply and demand (to the nearest unit) if baseball caps are \$4 each. Discuss the stability of the baseball cap market at this price level.

(B) Find the supply and demand (to the nearest unit) if baseball caps are \$9 each. Discuss the stability of the baseball cap market at this price level.

(C) Find the equilibrium price and quantity.

Regular midterm solutions

Problem 1

Part A

The revenue function is always $R(x) = xp(x)$ and the break-even points are those x where $R(x) = C(x)$, so

$$R(x) = xp(x) = x(27 - 3x) = 27x - 3x^2.$$

Break-even point x_b such that

$$27x_b - 3x_b^2 = 15x_b + 9$$

$$\begin{aligned} 27x_b - 3x_b^2 - 15x_b - 9 &= 0 \\ -3x_b^2 + 12x_b - 9 &= 0 \\ -3(x_b^2 - 4x_b + 3) &= 0 \\ x_b^2 - 4x_b + 3 &= 0 \\ (x_b - 1)(x_b - 3) &= 0. \end{aligned}$$

So $x_b = 1$ or $x_b = 3$.

Part B

Profit function is always $P(x) = R(x) - C(x)$. In this case it is

$$P(x) = 27x - 3x^2 - 15x - 9 = -3x^2 + 12x - 9.$$

It is a quadratic function i.e. of the form $ax^2 + bx + c$ with $a < 0$, so it attains the max at the vertex point $x_0 = \frac{-b}{2a}$.

$$x_0 = \frac{-12}{2 \cdot (-3)} = 2.$$

Hence, $P_{max} = P(x_0) = -3 \cdot 2^2 + 12 \cdot 2 - 9 = -12 + 24 - 9 = 3$

Part C

We know that the quantity that maximizes the profit is $x_0 = 2$, so the price that maximizes the profit is $p(x_0) = 27 - 3 \cdot 2 = 27 - 6 = 21$.

Problem 2

Part A

$$81(27^{x-2})^{x+1} = 3^{4x+4},$$

Note that $81 = 3^4$ and $27 = 3^3$, so we have

$$3^4((3^3)^{x-2})^{x+1} = 3^{4x+4},$$

$$3^{4+3(x-2)(x+1)} = 3^{4x+4}$$

or

$$4 + 3(x-2)(x+1) = 4x + 4.$$

FOIL to have

$$4 + 3x^2 - 3x - 6 = 4x + 4$$

$$3x^2 - 7x - 6 = 0.$$

Use quadratic formula to solve or factor as $(3x + 2)(x - 3) = 0$ to have $x = -2/3$ or $x = 3$.

Part B

$$\log_3(5 + 2x) - 1 = \frac{1}{2}\log_3 25$$

$$\log_3(5 + 2x) - 1 = \log_3(25)^{\frac{1}{2}}$$

$$\log_3(5 + 2x) - 1 = \log_3 5$$

$$\log_3(5 + 2x) - \log_3 5 = 1$$

$$\log_3 \frac{5 + 2x}{5} = 1$$

$$\frac{5 + 2x}{5} = 3^1$$

$$\frac{5 + 2x}{5} = 3$$

$$5 + 2x = 15$$

$$x = 5.$$

Plugging into the equation, we make sure that it is indeed a solution.

Part C

$$2018^{7x^2-68x+138} - 2018^{2(x^2-8)+x} = 0$$
$$2018^{7x^2-68x+138} = 2018^{2(x^2-8)+x}$$
$$7x^2 - 68x + 138 = 2(x^2-8) + x$$
$$7x^2 - 68x + 138 = 2x^2 - 16 + x$$
$$5x^2 - 69x + 154 = 0.$$

Use quadratic formula or factor as $(5x-14)(x-11) = 0$ to have $x = 11$ or $x = 2.8$.

Part D

$$\log_3(2x+3) + \log_3(x-1) = \log_3(4x-3)$$
$$\log_3(2x+3)(x-1) = \log_3(4x-3)$$
$$(2x+3)(x-1) = 4x-3$$
$$2x^2 + x - 3 = 4x - 3$$
$$2x^2 + x - 3 - 4x + 3 = 0$$
$$2x^2 - 3x = 0$$
$$x(2x-3) = 0$$

So $x = 3/2$ or $x = 0$. However, by plugging into the original equation, we see that $x = 0$ makes the second term on the left $\log_3(-1)$. Thus, the only solution is $x = 3/2$.

Problem 3

Part A

For an arithmetic sequence $a_n = a_1 + (n-1)d$. We have $a_13 = -53$, $a_{38} = -128$. So

$$-53 = a_1 + 12d$$

and

$$-128 = a_1 + 37d.$$

Subtracting one from another,

$$-128 - (-53) = a_1 + 37d - (a_1 + 12d)$$

or

$$-75 = 25d$$
$$d = -3.$$

Plugging this into the first equation we have $a_1 = -17$. Therefore,

$$a_{668} = a_1 + 667d = -17 + 667(-3) = -2018.$$

Part B

For geometric sequences, we have $b_n = b_1 r^{n-1}$ and $S_\infty = \frac{b_1}{1-r}$. So

$$6 = \frac{5}{1-r}$$
$$1 - r = \frac{5}{6}$$
$$r = 1 - \frac{5}{6}$$
$$r = \frac{1}{6}$$

and

$$b_3 = b_1 r^2 = 5\frac{1}{6^2} = \frac{5}{36}.$$

Problem 4

Part A

First Jeremy makes equal $PMT = 2500$ which earned him $i = \frac{0.064}{1} = 0.064$ annually. He made $n = 26$ deposits. So right after his last deposit (when he is 48) he had the amount

$$FV = PMT\frac{(1+i)^n - 1}{i} = 2500\frac{(1.064)^{26} - 1}{0.064} \approx 156931.3128$$

This money kept growing without additional payments, at the same rate. So in 16 more years, when Jeremy retires the balance is

$$A = P(1+i)^n = 156931.3128(1 + 0.064)^{16} \approx 423424.180031$$

Part B

The interest earned over the last six years is the difference between the balance at the end and the balance 6 years before the end (when Jeremy is 58 years old). We know the final balance from Part A. The balance when he is 58 is the balance 10 years after Jeremy stopped making payments. So it can be calculated as

$$A = P(1+i)^n = 156931.3128(1+0.064)^{10} \approx 291827.28$$

Hence the interest earned is

$$I = 423424.18 - 291827.28 = 131596.90$$

Problem 5

Part A

This is a loan payment so we use $PV = PMT\frac{1-(1+i)^{-n}}{i}$ formula. We have $PV = 151900$, $i = \frac{0.087}{12} = 0.00725$ and $n = 12 * 20 = 240$. Plugging in the known components, we have the monthly payments

$$PMT = \frac{iPV}{1-(1+i)^{-n}} = \frac{0.00725 \cdot 151900}{1-(1.00725)^{-240}} \approx \underline{1337.5143}$$

The total interest paid is the difference between how much money the family paid to the bank and how much they received from it. They received 151900 and they have been paying 1337.51 each of 240 months, i.e. $1337.51 \cdot 240 \approx 321003.44$ in total . So the interest paid is

$$I = 321003.44 - 151900 = \underline{169103.44}$$

Part B

If they decided to add 100 dollars to the monthly payments, their new PMT will be $PMT = 1337.51 + 100 = 1437.51$. Now they will cover the loan sooner than in 20 year. We can calculate how soon using $PV = PMT\frac{1-(1+i)^{-n}}{i}$.

$$151900 = 1437.51\frac{1-(1.00725)^{-n}}{0.00725}$$

$$\frac{151900 \cdot 0.00725}{1437.51} = 1-(1.00725)^{-n}$$

$$(1.00725)^{-n} = 1 - \frac{151900 \cdot 0.00725}{1437.51}$$
$$(1.00725)^{-n} = 1 - 0.76609901844 = 0.23390098155$$
$$\ln(1.00725)^{-n} = \ln 0.23390098155$$
$$-n(\ln 1.00725) = \ln 0.23390098155$$
$$n = -\frac{\ln 0.23390098155}{\ln 1.00725}$$
$$n \approx 201.12 \text{ months}$$

This is approximately $\underline{t = 17}$ years.

To find the interest saved one can consider the difference between the monthly payments of 1337.51 dollars for 240 months and monthly payments of 1437.51 dollars for 201 months

$$I = 1337.51 \cdot 240 - 1437.51 \cdot 201 = \underline{32062.89}$$

Problem 6

Supply-price relation is

$$p = 0.4q + 3.2$$

Demand-price relation is

$$p = -1.9q + 17$$

Part A

Plug in $p = 4$ to the supply-price equation to have

$$4 = 0.4q + 3.2$$

$$q = \frac{4 - 3.2}{0.4} = 2 \text{ (the corresponding supply in hundreds)}$$

Plug in $p = 4$ to the demand-price equation to have

$$4 = -1.9q + 17$$

$$q = \frac{4 - 17}{-1.9} \approx 6.84 \text{ (the corresponding demand in hundreds)}$$

Demand exceeds supply, so the market is unstable.

Part B

Plug in $p = 9$ to the supply-price equation to have

$$9 = 0.4q + 3.2$$

$$q = \frac{9 - 3.2}{0.4} = 14.5 \text{ (the corresponding supply in hundreds)}$$

Plug in $p = 4$ to the demand-price equation to have

$$9 = -1.9q + 17$$

$$q = \frac{9 - 17}{-1.9} \approx 4.21 \text{ (the corresponding demand in hundreds)}$$

Supply exceeds demand, so the market is unstable.

Part C

Equilibrium price is the one which equates the supply and demand. For such price there are q_e hundreds caps such that

$$0.4q_e + 3.2 = -1.9q_e + 17$$

$$2.3q_e = 13.8$$

$$q_e = \frac{13.8}{2.3} = 6 \text{ (equilibrium quantity in hundreds).}$$

For the equilibrium price use either or the equations. E.g. the price-supply to have

$$p = 0.4q_e + 3.2 = 0.4 \cdot 6 + 3.2 = 2.4 + 3.2 = 5.6 \text{ (equilibrium price)}$$