VERSION 11

UNIVERSITY OF CALGARY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS

MIDTERM TEST 1 (SOLUTION)

MATH 265 L01-L05 Fall 2017

DATE: 19 October 2017 **Time: 7:00 pm – 8:30 pm**

Student ID Number:	Last Name:	First Name:	Lecture Section:

EXAMINATION RULES

1. This is a closed book examination.
2. No calculators and no other aids are allowed for this examination
3. Answer Part A on the scantron sheet and Part B on the examination paper
4. Scantron sheets must be filled out during the exam time limit. No additional time will be granted to fill in scantron form.
5. The use of personal electronic or communication devices is prohibited.
6. A University of Calgary Student ID card is required to write the Final Examination and could be requested for midterm examinations. If adequate ID isn't present the student must complete an Identification Form.
7. Students late in arriving will not be permitted after one-half hour of the examination time has passed.
8. No student will be permitted to leave the examination room during the first 30 minutes, nor during the last 15 minutes of the examination. Students must stop writing and hand in their exam immediately when time expires.
9. All inquiries and requests must be addressed to the exam supervisor.
10. Students are strictly cautioned against:
    a. communicating to other students;
    b. leaving answer papers exposed to view;
    c. attempting to read other students' examination papers
11. During the final examination, if a student becomes ill or receives word of domestic affliction, the student must report to the Invigilator, hand in the unfinished paper and request that it be cancelled. If ill, the student must report immediately to a physician/counselor for a medical note to support a deferred examination application.
12. Once the examination has been handed in for marking, a student cannot request that the examination be cancelled. Retroactive withdrawals from the course will be denied.
13. Failure to comply with these regulations will result in rejection of the examination paper.

Question	Total Value	Actual Marks
Part A	70	
15	5	
16	15	
17	10	
Part C	10	
Total	100	

PART (A) Multiple Choice Questions (70 marks)

Instructions: Put your answers in the scantron sheet for grading. Circle your answers in your test paper for your reference. Each question is worth 5 marks.

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1. What is the value of

$$\lim_{x\to 16} \frac{x - 2\sqrt{x} - 8}{\sqrt{x} - 4} \ ?$$

[Hint: Use a new variable $t = \sqrt{x}$.]

(a) 0

(b) 2

(c) 4

(d) ***6

(e) 8

[Solution. Let $t = \sqrt{x}$. Then $x = t^2$, and $t \to 4$ if and only if $x \to 16$. So,

$$\lim_{x\to 16} \frac{x - 2\sqrt{x} - 8}{\sqrt{x} - 4} = \lim_{t\to 4} \frac{t^2 - 2t - 8}{t - 4} = \lim_{t\to 4} \frac{(t-4)(t+2)}{t-4} = \lim_{t\to 4} (t+2) = 6.]$$

2. Let $f(x) = 2^x - \frac{1}{x}$. Which of the following is a correct argument that proves that the equation $f(x) = 0$ has a root between 0 and 1?

(a) We have that $f(0) < 0$ and $f(1) > 0$ and that f is continuous on $[0, 1]$. Thus, by the Intermediate Value Theorem, there is a number in $[0, 1]$ such that $f(c) = 0$.

(b) We have that $f(0) < 0$ and $f(1) = -1 < 0$ and that f is continuous on $[0, 1]$. Thus, by the Intermediate Value Theorem, there is a number in $[0, 1]$ such that $f(c) = 0$.

(c) We have that $f(0) = 1 > 0$ and $f(1) = -1 < 0$ and that f is continuous on $[0, 1]$. Thus, by the Intermediate Value Theorem, there is a number in $[0, 1]$ such that $f(c) = 0$.

(d) We have that $f(0.1) = 2^{0.1} - 1 > 0$ and $f(1) = 1 > 0$ and that f is continuous on $[0.1, 1]$. Thus, by the Intermediate Value Theorem, there is a number in $[0.1, 1]$, and hence in $[0, 1]$, such that $f(c) = 0$.

(e) ***We have that $f(0.1) = 2^{0.1} - 10 < 0$ and $f(1) = 1 > 0$ and that f is continuous on $[0.1, 1]$. Thus, by the Intermediate Value Theorem, there is a number in $[0.1, 1]$, and hence in $[0, 1]$, such that $f(c) = 0$.

[Solution. The function $f(x)$ is undefined at $x = 0$. So, answer choices (a), (b) and (c) are all incorrect. We cannot evaluate the function at $x = 0$. Answer choices (d) and (e) consider the value of $f(x)$ at some x slightly larger than 0, specifically, $x = 0.1$. Since

$$f(0.1) = 2^{0.1} - \frac{1}{0.1} = 2^{0.1} - 10,$$

this eliminates (d), and (e) is the only answer that can possibly be correct. Even if we do not know how to get $\frac{1}{0.1} = 10$, we should be able to tell that answer choice (d) is incorrect. For (d) starts with $f(0, 1) > 0$ and $f(1) > 0$, where 0 does not lie between the two function values. So, the condition for the Intermediate Value Theorem is not satisfied. The conclusion that there is a root between 0.1 and 1 is not warranted. For the sake of completeness, we should check that (e) is indeed a correct argument. First, the calculations are correct, and second, it is true that $2^{0.1} - 10 < 0$ and $1 > 0$. So, 0 does lie between the function values at 0.1 and 1. Third, the function is indeed continuous on $[0.1, 1]$. So, the Intermediate Value Theorem applies and we can conclude that there is a root between 0.1 and 1. Finally, being between 0.1 and 1 the root must be between 0 and 1.]

3. If we apply the method of logarithmic differentiation to find the derivative of

$$f(x) = \frac{(x^2+1)^2 (e^x+4)}{\sqrt{x}},$$

we will arrive at an answer that looks like

$$f'(x) = \frac{(x^2+1)^2 (e^x+4)}{\sqrt{x}} \times \text{some function } g(x).$$

What is $g(x)$?

(a) $\frac{2}{x^2+1} + \frac{1}{e^x+4} - \frac{1}{\sqrt{x}}$

(b) $\frac{4x}{x^2+1} + \frac{e^x}{e^x+4} - \frac{1}{\sqrt{x}}$

(c) $\frac{2}{x^2+1} + \frac{1}{e^x+4} - \frac{1}{2x}$

(d) ***$\frac{4x}{x^2+1} + \frac{e^x}{e^x+4} - \frac{1}{2x}$

[Solution. We apply ln to both sides and simplify:

$$\begin{aligned} \ln f(x) &= \ln\left(\frac{(x^2+1)^2 (e^x+4)}{\sqrt{x}}\right) \\ &= \ln\left((x^2+1)^2 (e^x+4)\right) - \ln\sqrt{x} \\ &= \ln\left((x^2+1)^2\right) + \ln(e^x+4) - \ln\left(x^{1/2}\right) \\ &= 2\ln\left(x^2+1\right) + \ln(e^x+4) - \frac{1}{2}\ln x. \end{aligned}$$

Differentiating both sides with respect to x and using the chain rule, weget

$$\begin{aligned} \frac{f'(x)}{f(x)} &= 2\left(\frac{2x}{x^2+1}\right) + \frac{e^x}{e^x+4} - \frac{1}{2}\frac{1}{x} \\ &= \frac{4x}{x^2+1} + \frac{e^x}{e^x+4} - \frac{1}{2x}. \end{aligned}$$

Multiplying both sides by $f(x)$, we get

$$\begin{aligned} f'(x) &= f(x)\left(\frac{4x}{x^2+1} + \frac{e^x}{e^x+4} - \frac{1}{2x}\right) \\ &= \frac{(x^2+1)^2 (e^x+4)}{\sqrt{x}} \times \left(\frac{4x}{x^2+1} + \frac{e^x}{e^x+4} - \frac{1}{2x}\right).] \end{aligned}$$

4. Suppose that y is a differentiable function of x which satisfies the equation

$$\cos(xy) = 3x + 2.$$

Which of the following is equal to $\frac{dy}{dx}$?

(a) $\dfrac{-3}{\sin(xy)}$

(b) ***$\dfrac{3 + y\sin(xy)}{-x\sin(xy)}$

(c) $\dfrac{3 - y\sin(xy)}{x\sin(xy)}$

(d) $\dfrac{3 + x\sin(xy)}{-y\sin(xy)}$

(e) $\dfrac{3 - x\sin(xy)}{y\sin(xy)}$

[Solution. Differentiating the given equation with respect to x and using the chain rule and the product rule, we get

$$\begin{aligned}
-\sin(xy) \cdot \left(1y + x\frac{dy}{dx}\right) &= 3, \\
-y\sin(xy) - x\sin(xy)\frac{dy}{dx} &= 3, \\
-x\sin(xy)\frac{dy}{dx} &= 3 + y\sin(xy), \\
\frac{dy}{dx} &= \frac{3 + y\sin(xy)}{-x\sin(xy)}.]
\end{aligned}$$

5. Consider the function
$$f(x) = \frac{x^2 - x}{x^2 - 1}.$$
Which of the following is the **exact list** of horizontal and vertical asymptotes of f?

(a) ***The vertical asymptote is $x = -1$; the horizontal asymptote is $y = 1$.

(b) The vertical asymptotes are $x = -1$ and $x = 1$; the horizontal asymptote is $y = 1$.

(c) The vertical asymptotes are $x = 0$, $x = 1$ and $x = -1$; the horizontal asymptote is $y = 1$.

(d) The vertical asymptote is $x = -1$; the horizontal asymptotes are $y = 1$ and $y = -1$.

(e) The horizontal asymptote is $x = -1$; the vertical asymptote is $y = 1$.

[Solution. Vertical asymptotes occur exactly where one or both of the one-sided limits is either ∞ or $-\infty$. In the case of rational functions, this can only occur where the denominator is zero. For at all other points, the function is continuous and the limit will be finite. Here, the denominator is zero exactly when $x = -1$ or 1. So, there are the only possible places where the function can have a vertical asymptote. We check the (one-sided) limits at these points. We note that
$$\frac{x^2 - x}{x^2 - 1} = \frac{x(x-1)}{(x+1)(x-1)} = \frac{x}{x+1}.$$
This shows that the limit as $x \to 1$ is finite. So, $x = 1$ is not a vertical asymptote. On the other hand, $f(x) \to -\infty$ as $x \to -1^+$ (and $f(x) \to +\infty$ as $x \to -1^-$). So, $x = -1$ is a vertical asymptote. So, there is exactly one vertical asymptote, namely, $x = -1$. This eliminates answer choices (b), (c) and (e). For horizontal asymptote(s), we look at the behaviour of $f(x)$ at the far right and the far left. So, we calculate the limits as $x \to +\infty$ and $x \to -\infty$. Now, dividing both the top and the bottom by x, we get
$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x}{x+1} = \lim_{x\to+\infty} \frac{\frac{x}{x}}{\frac{x}{x}+\frac{1}{x}} = \lim_{x\to+\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = 1,$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{x+1} = \lim_{x\to-\infty} \frac{\frac{x}{x}}{\frac{x}{x}+\frac{1}{x}} = \lim_{x\to-\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = 1.$$
So, at both the far right and the far left, $f(x)$ approaches 1. So, there is exactly one horizontal asymptote, namely, $y = 1$. So, (a) is the correct answer.]

6. Consider the function

$$f(x) = xe^{2x}.$$

Routine calculations show that

$$f'(x) = (1+2x)\,e^{2x} \text{ and } f''(x) = 4(1+x)\,e^{2x}.$$

Which of the following best describes the open interval(s) on which f is increasing?

(a) $(-\infty, \infty)$

(b) *** $\left(-\frac{1}{2}, \infty\right)$

(c) $\left(-\infty, -\frac{1}{2}\right)$

(d) $(-1, \infty)$

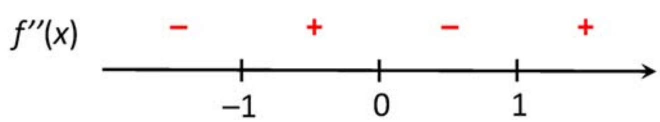
(e) $(-\infty. -1)$

[Solution. The intervals of increase and decrease are determined by the sign of $f'(x)$. In particular, $f''(x)$ is irrelevant to this question. Since e^{2x} is positive for all x, the sign of $f'(x)$ is the same as that of $1+2x$, which is positive when $x > -\frac{1}{2}$ and negative when $x < -\frac{1}{2}$. So, f is increasing on the open interval $\left(-\frac{1}{2}, \infty\right)$ and decreasing on the open interval $\left(-\infty, -\frac{1}{2}\right)$.]

7. Consider the function
$$f(x) = \frac{x^3}{6} + x - x\ln|x|, \ x \neq 0.$$
Routine calculations show that for any $x \neq 0$,
$$f'(x) = \frac{x^2}{2} - \ln|x| \qquad \text{and} \qquad f''(x) = x - \frac{1}{x}.$$
Which of the following correctly describes the open intervals on which f is concave downward?

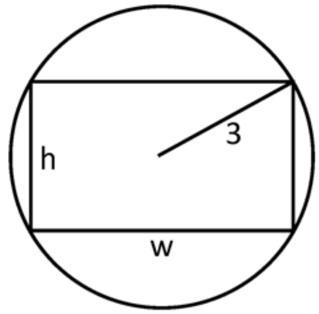
(a) ***$(-\infty, -1)$ and $(0, 1)$
(b) $(-1, 0)$ and $(1, \infty)$
(c) $(-\infty, -1)$ and $(1, \infty)$
(d) $(-1, 0)$ and $(0, 1)$
(e) $(-1, 1)$

[Solution. Concavity is determined by the sign of the second derivative $f''(x)$. In particular, the information about $f'(x)$ is not needed. To analyze the sign of $f''(x)$, we need to factorize it completely:
$$f''(x) = x - \frac{1}{x} = \frac{x^2 - 1}{x} = \frac{(x+1)(x-1)}{x}.$$
Either using a table or a sign-diagram,



we see that $f''(x)$ is positive on the intervals $(1, \infty)$ and $(-1, 0)$ and is negative on the intervals $(0, 1)$ and $(-\infty, -1)$. So, $f(x)$ is concave upward on the intervals $(1, \infty)$ and $(-1, 0)$ and is concave downward on the intervals $(0, 1)$ and $(-\infty, -1)$.]

8. A beam with a rectangular cross-section is cut from a cylindrical log of radius 3 cm. The strength of the beam is proportional to wh^2, where w is the width and h the height of its cross-section.



What value of w will give the beam maximum strength?

(a) $w = \sqrt{3}$ cm
(b) $w = \sqrt{6}$ cm
(c) ***$w = \sqrt{12}$ cm
(d) $w = \sqrt{15}$ cm
(e) $w = \sqrt{18}$ cm

[Solution. We note that the diameter is 6 cm. So, by Pythagorus' Theorem, we have

$$w^2 + h^2 = 6^2,$$

which implies that $h^2 = 36 - w^2$. The strength function is, thus,

$$S(w) = wh^2 = w(36 - w^2) = 36w - w^3.$$

It is clear from the physical situation that $0 \le w \le 6$. So, we now find the absolute maximum for $S(w)$. Differentiating and setting to zero, we find the critical point(s):

$$S'(w) = 36 - 3w^2 = 3(12 - w^2) = 0 \iff w^2 = 12 \iff w = \sqrt{12},$$

since the negative square root is outside the domain of $S(w)$. There are several ways to determine that this value of $w = \sqrt{12}$ gives us the absolute maximum for $S(w)$. Any one of these arguments will work:

Method 1: We compare the function value at the critical point and the endpoints. Since

$$S(0) = S(6) = 0 \text{ and } S\left(\sqrt{12}\right) > 0,$$

the absolute maximum occurs at $w = \sqrt{12}$.

Method 2: We analyze the sign of the derivative

$$S'(w) = 3\left(\sqrt{12} + w\right)\left(\sqrt{12} - w\right)$$

and see that $S'(w) > 0$ when $0 < w < \sqrt{12}$ and $S'(w) < 0$ when $\sqrt{12} < w < 6$. So, $S(w)$ increases on $\left[0, \sqrt{12}\right]$, reaches a highest point at $w = \sqrt{12}$ and then decreases on $\left[\sqrt{12}, 6\right]$. This proves that the absolute maximum occurs at $w = \sqrt{12}$.

Method 3: We calculate the second derivative

$$S''(w) = -6w < 0 \text{ for all } w \text{ in the interval } (0, 6).$$

This means that the graph of $S(w)$ is concave downward on the interval $(0, 6)$. The lone critical point $w = \sqrt{12}$ must corresponds to the absolute maximum.]

9. Let $y = f(x)$ be differentiable on some open interval containing 3, $f(3) = 7$ and $f'(3) = 2$. If we now increase the value of x from 3 to 3.2, which of the following best describes the impact on the value of y?

   (a) The value of y will be increased by an amount approximately equal to 0.2, thus y will be increased from 7 to approximately 7.2.

   (b) The value of y will be decreased by an amount approximately equal to 0.2, thus y will be decreased from 7 to approximately 6.8.

   (c) ***The value of y will be increased by an amount approximately equal to 0.4, thus y will be increased from 7 to approximately 7.4.

   (d) The value of y will be decreased by an amount approximately equal to 0.4, thus y will be decreased from 7 to approximately 6.6.

   (e) The value of y may be increased or decreased, but it is still approximately equal to 7.

   [Solution. We could analyze this by either considering the definition of the derivative or by using linear approximation - remember, linear approximation is another way of expressing differentiability. If we approach it by considering the defintion of the derivative, then we would argue as follows:

$$f'(3) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \Longrightarrow f'(3) \approx \frac{\Delta y}{\Delta x} \text{ when } \Delta x \text{ is a small change from } x = 3.$$

   Now, $f'(3) = 2$ and $\Delta x = 3.2 - 3 = 0.2$. So,

$$\Delta y \approx f'(3)\,\Delta x = (2)(0.2) = 0.4.$$

   So, y is increased by an amount approximately equal to 0.4. Consequently, y will increase from $y = 7$ to approximately $y = 7.4$.
   If we use linear approximation, then we would argue somewhat like this:

$$f(x) \approx f(3) + f'(3)(x-3) = 7 + 2(x-3) \text{ for } x \text{ near } 3.$$

   Therefore,

$$f(3.2) \approx 7 + 2(3.2 - 3) = 7 + 2(0.2) = 7 + 0.4 = 7.4.$$

   So, y is increased by an amount approximately equal to 0.4, from $y = 7$ to approximately $y = 7.4$.]

10. Suppose that f is differentiable at 2, $f(2) = 3$ and $f'(2) = -5$. Which of the following statement is **not** necessarily true?

    (a) f is continuous at 2

    (b) ***$f'(x)$ exists for all x sufficiently close to 2 and

    $$\lim_{x\to 2} f'(x) = -5.$$

    (c) If $E(x) = |f(x) - 3|$ denotes the error of the approximation of $f(x)$ by the constant function $y = 3$ near $x = 2$, then

    $$\lim_{x\to 2} E(x) = 0.$$

    (d) If $E(x) = |f(x) - L_2(x)|$ denotes the error of the approximation of $f(x)$ by its linearization $L_2(x) = 3 - 5(x - 2)$ at 2, then

    $$\lim_{x\to 2} E(x) = 0.$$

    (e) If $E(x) = |f(x) - L_2(x)|$ denotes the error of the approximation of $f(x)$ by its linearization $L_4(x) = 3 - 5(x - 2)$ at 2, then

    $$\lim_{x\to 2} \frac{E(x)}{|x-2|} = 0.$$

    [Solution. Since f is differentiable at 2, it is continuous at 2. Statement (a) is true. It follows that $E(x) = |f(x) - 3| = |f(x) - f(2)| \to 0$ as $x \to 2$. So, statement (c) is also true. Differentiability implies statement (e). It is a theorem. In particular, statement (d) is also true, since it follows logically from statement (e): If $E(x)$ is as in (d) and (e), then

    $$\lim_{x\to 2} E(x) = \lim_{x\to 2} \frac{E(x)}{|x-2|} |x - 2| = 0 \cdot 0 = 0.$$

    Only statement (b) is not necessarily true. Statement (b) claims that the derivative function f' is continuous at 2. This is not necessarily if all it is given is that f' exists at 2. This is no guarantee that $f'(x)$ even exists for x near 2.

11. Suppose we apply Newton's Method to the function $f(x) = x^3 - x^2 - 3$ to find a numerical solution of the equation $x^3 - x^2 - 3 = 0$. If we start with the first estimate $x_1 = 2$, what is the value of x_2?

    (a) 1

    (b) $\frac{1}{2}$

    (c) $\frac{3}{4}$

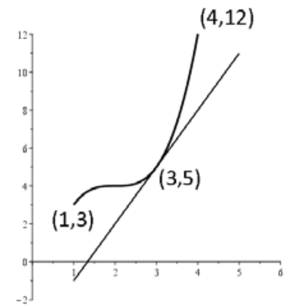
    (d) ***$\frac{15}{8}$

    (e) $\frac{31}{16}$

    [Solution. We use Newton's recurson formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^3 - x_1^2 - 3}{3x_1^2 - 2x_1} = 2 - \frac{2^3 - 2^2 - 3}{3(2^2) - 2(2)} = 2 - \frac{1}{8} = \frac{15}{8}.]$$

12. The following graph shows a function for which the Mean Value Theorem applies on the interval $[1,4]$. If it turns out that $c=3$ satisfies the conclusion of the Mean Value Theorem, what is the equation of the tangent line at the point $(3,5)$?



(a) ***$y=3x-4$

(b) $y=3x+5$

(c) $y=x+2$

(d) $y=4x-7$

(e) $y=4x+5$

[Solution. Since $c=3$ satisfies the conclusion of the Mean Value Theorem on the interval $[1,4]$, we have

$$f'(3)=\frac{f(4)-f(1)}{4-1}=\frac{12-3}{4-1}=\frac{9}{3}=3.$$

So, the equation of the tangent line at the point $(3,5)$ is

$$y=f(3)+f'(3)(x-3)=5+3(x-3)=3x-4.]$$

13. Suppose that f is a differentiable function and that $-3 \leq f'(x) \leq 4$ for all x. If $f(2) = -1$, what is the largest possible value for $f(5)$?

    (a) -10

    (b) 4

    (c) 8

    (d) 10

    (e) ***11

    [Solution. By the Mean Value Theorem, there is some c in $(2, 5)$ such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2} = \frac{f(5) - (-1)}{3} = \frac{f(5) + 1}{3}.$$

By what is given about $f'(x)$, we have $-3 \leq f'(c) \leq 4$. So,

$$\begin{aligned} -3 &\leq \frac{f(5) + 1}{3} \leq 4, \\ -9 &\leq f(5) + 1 \leq 12 \\ -10 &\leq f(5) \leq 11. \end{aligned}$$

So, $f(5)$ cannot exceed 11. We also note that $f(5)$ could equal 11. If f is the linear function with slope 4 and passing through the point $(2, -1)$, then it will pass through the point $(5, 11)$. So, 11 is the largest possible value for $f(5)$.]

14. Find the degree 2 Taylor polynomial of $f(x) = e^{5x}$ about $x = 0$.

(a) $1 + 5x$

(b) $1 + 5x + 5x^2$

(c) $1 + 5x + \frac{5}{2}x^2$

(d) $1 + 5x + 25x^2$

(e) ***$1 + 5x + \frac{25}{2}x^2$

[Solution. First, $f(0) = e^0 = 1$. Then,

$$\begin{aligned} f'(x) &= 5e^{5x} \Longrightarrow f'(0) = 5, \\ f''(x) &= 25e^{5x} \Longrightarrow f''(0) = 25. \end{aligned}$$

So, the degree 2 Taylor polynomial is

$$\begin{aligned} T_2(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 \\ &= 1 + 5x + \frac{25}{2}x^2.] \end{aligned}$$

PART (B): Written Answer Questions (30 marks).

Instructions: Answer Questions 15-17 in the space provided. Be sure to show all of your work, as partially correct answers may be worth partial credit. Please put your answer in the answer box.

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15. (5 marks) Use the Squeeze Theorem to prove that

$$\lim_{x\to+\infty}\frac{1}{x+e^{-x}}=0.$$

[Hint: For $x \geq 0$, $0 < e^{-x} \leq 1$.]
[Solution. For $x \geq 0$, $0 < e^{-x} \leq 1$. So, adding x and then taking reciprocal for $x > 0$,

$$\begin{aligned} x &< x+e^{-x} \leq x+1, \\ \frac{1}{1+x} &\leq \frac{1}{x+e^{-x}} < \frac{1}{x}. \end{aligned}$$

Since

$$\lim_{x\to+\infty}\frac{1}{x+1}=\lim_{x\to+\infty}\frac{1}{x}=0,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to+\infty}\frac{1}{x+e^{-x}}=0.]$$

16. Apply suitable rules to write down the derivatives of the following functions. Do **not** simplify your answers.

   (a) (5 marks) $f(x) = \sin^5 x \cos^4 x$
   [Solution. By the product rule and the chain rule,

   ANSWER: $f'(x) = \left(5\sin^4 x \cos x\right) \cdot \cos^4 x + \sin^5 x \cdot \left(4\cos^3 x \cdot (-\sin x)\right).$
   (If simplified one step: $= 5\sin^4 x \cos^5 x - 4\sin^6 x \cos^3 x$)

   (b) (5 marks) $f(x) = \ln\left(x^{2/3} + e^{-x}\right)$
   [Solution. By the chain rule (twice),

   ANSWER: $f'(x) = \dfrac{1}{x^{2/3} + e^{-x}} \left(\dfrac{2}{3}x^{-1/3} + e^{-x} \cdot (-1)\right).$

   (c) (5 marks) $f(x) = \dfrac{\tan(2x)}{1 + \sec(2x)}$
   [Solution. By the quotient rule and the chain rule,

   ANSWER: $f'(x) = \dfrac{\left(2\sec^2(2x)\right) \cdot \left(1 + \sec(2x)\right) - \left(\tan(2x)\right) \cdot \left(2\sec(2x)\tan(2x)\right)}{\left(1 + \sec(2x)\right)^2}$

17. Consider the function $g(x) = \sqrt{x}$.

(a) (7 marks) Find the local linearization $L_1(x)$ of $g(x)$ at $x = 1$ and use it to estimate the value of $\sqrt{1.1}$.

[Solution. We have $g(1) = \sqrt{1} = 1$ and $g'(x) = \dfrac{1}{2\sqrt{x}} \Longrightarrow g'(1) = \dfrac{1}{2}$. So, the local linearization at $x = 1$ is

$$L_1(x) = g(1) + g'(1)(x-1) = 1 + \frac{1}{2}(x-1).$$

Therefore, we have the linear approximation

$$\sqrt{x} \approx 1 + \frac{1}{2}(x-1) \text{ for } x \text{ near } 1.$$

In particular,

$$\sqrt{1.1} \approx 1 + \frac{1}{2}(1.1-1) = 1 + 0.05 = 1.05 \left(\text{or, } 1 + \frac{1}{20} = \frac{21}{20}\right)$$

ANSWER: $L_1(x) = 1 + \dfrac{1}{2}(x-1)$

ANSWER: $\sqrt{1.1} \approx 1.05$, or, $\dfrac{21}{20}$.

(b) (3 marks) If a function f is differentiable at a, then the error of linear approximation $E(x) = |f(x) - L_a(x)|$ for x close to a can be estimated by

$$E(x) \approx \frac{|f''(a)|}{2}(x-a)^2.$$

Use this to estimate how good the estimate you obtained in part (a) is.

[Solution. For $g(x) = \sqrt{x}$, we have

$$g'(x) = \frac{1}{2}x^{-1/2} \text{ and } g''(x) = \frac{1}{2}\left(-\frac{1}{2}x^{-3/2}\right) = \frac{-1}{4x^{3/2}}.$$

So, $g''(1) = \dfrac{-1}{4}$. Therefore, the error of the approximation at $x = 1.1$ is

$$E(1.1) \approx \frac{\|g''(1)\|}{2}(1.1-1)^2 = \frac{1}{8}(0.01) = \frac{1}{800} = 0.00125.$$

ANSWER: The error of the estimate we obtained in part (a) is approximately equal to $\dfrac{1}{800}$.

PART (C): Bonus Questions (Up to 10 marks)

Answer ANY number of the following questions. You can earn up to 5 bonus marks for each question for a maximum total of 10 bonus mark from this section.

18. Prove directly from the definition of limit (i.e., using an argument with ε and δ) that

$$\lim_{x \to 1} (5x + 2) = 7.$$

[Solution. For any $\varepsilon > 0$, let $\delta = \dfrac{\varepsilon}{5} > 0$. Then, if x lies within δ units of 1 but is not equal to 1, we have

$$\begin{aligned} 1 - \delta &< x < 1 + \delta, \\ 1 - \frac{\varepsilon}{5} &< x < 1 + \frac{\varepsilon}{5} \text{ (by the definition of } \delta) \\ 5 - \varepsilon &< 5x < 5 + \varepsilon \text{ (by multiplying the last line by 5)} \\ 7 - \varepsilon &< 5x + 2 < 7 + \varepsilon \text{ (by adding 2 to the last line)} \end{aligned}$$

i.e., $5x + 2$ lies within ε units of 7. Therefore, by definition,

$$\lim_{x \to 1} (5x + 2) = 7.$$

Alternatively, one could argue using absolute value after δ has been chosen as above. The argument would go like this: If x is any number such that $|x - 1| < \delta$ and $x \neq 1$, then

$$|(5x + 2) - 7| = |5x - 5| = 5\,|x - 1| < 5\delta = \varepsilon.$$

Therefore, by definition,

$$\lim_{x \to 1} (5x + 2) = 7.]$$

19. Let $f(x) = x^2$ and let $L_3(x) = 9+6(x-3)$ be the linearization of $f(x)$ at $x = 3$. Verify that as $x \to 3$, the error of linear approximation $E(x) = |f(x) - L_3(x)|$ approaches zero faster than does $|x-3|$. That is, verify that

$$\lim_{x\to 3} \frac{E(x)}{|x-3|} = 0.$$

Alternatively, you can prove this fact in general for an arbitrary function f differentiable at a :

$$\lim_{x\to a} \frac{E(x)}{|x-a|} = 0,$$

where $E(x) = |f(x) - L_a(x)|$.
[Solution. We first simplify $E(x)$ as follows:

$$E(x) = |f(x) - L_3(x)| = \left|x^2 - 9 - 6(x-3)\right| = \left|x^2 - 6x + 9\right| = \left|(x-3)^2\right| = |x-3|^2.$$

So,

$$\lim_{x\to 3} \frac{E(x)}{|x-3|} = \lim_{x\to 3} \frac{|x-3|^2}{|x-3|} = \lim_{x\to 3} |x-3| = 0.$$

This verifies the claim for the given function at the given point. A proof of the general case would go like this:

$$\begin{aligned}
\lim_{x\to a} \frac{E(x)}{|x-a|} &= \lim_{x\to a} \frac{|f(x) - L_a(x)|}{|x-a|} = \lim_{x\to a} \left|\frac{f(x) - L_a(x)}{x-a}\right| \\
&= \lim_{x\to a} \left|\frac{f(x) - f(a) - f'(a)(x-a)}{x-a}\right| = \lim_{x\to a} \left|\frac{f(x) - f(a)}{x-a} - \frac{f'(a)(x-a)}{x-a}\right| \\
&= \lim_{x\to a} \left|\frac{f(x) - f(a)}{x-a} - f'(a)\right| = |f'(a) - f'(a)| = 0.]
\end{aligned}$$

20. Prove that for any real numbers x and y,

$$|\sin x - \sin y| \leq |x - y|.$$

[Hint: Mean Value Theorem.]
[Solution. If $x = y$, then both sides are equal to zero and so, the inequality holds. So, we assume that $x \neq y$. Let I be the closed interval whose endpoints are x and y. Then, since the sine function is everywhere differentiable, the function $f(t) = \sin t$ is continuous on I and differentiable on the interior of I. By the Mean Value Theorem, there is some c in the interior of I such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}.$$

Since $f'(t) = \cos t$, we have

$$f(x) - f(y) = f'(c)(x - y) = (\cos c)(x - y).$$

Taking absolute value and using the fact that $|\cos c| \leq 1$, we have

$$|f(x) - f(y)| = |(\cos c)(x - y)| = |\cos c| |x - y| \leq 1 \cdot |x - y| = |x - y|.$$

The proof is complete.

21. Use the Racetrack Principle to prove that for any $x > 0$,

$$e^x > 1 + x.$$

[Solution. For any arbitrary but fixed $x > 0$, let I denote the closed interval $[0, x]$. Consider the functions $f(t) = e^t$ and $g(t) = 1 + t$. Then, f and g are continuous on $[0, x]$ and differentiable on $(0, x)$. We have $f(0) = 1 = g(0)$, and since

$$f'(t) = e^t > e^0 = 1 = g'(t)$$

for all $t \in (0, x)$, it follows from the Racetrack principle that $f(x) > g(x)$, i.e., $e^x > 1 + x$.]

22. One version of the Constant Function Theorem says that if $f'(x) = 0$ for all x in an interval, then $f(x)$ is a constant function on that interval. Use this to prove that if $g(x)$ is a function such that
$$g'(x) = g(x)$$
for all real numbers x, then there is some constant C such that $g(x) = Ce^x$. [Hint: Consider the derivative of the function $f(x) = e^{-x}g(x)$.]
[Solution. Let $f(x) = e^{-x}g(x)$. Then by the product rule and the chain rule, we have
$$f'(x) = \left(-e^{-x}\right) \cdot g(x) + e^{-x} \cdot g'(x) = -e^{-x}g(x) + e^{-x}g(x) = 0$$
for all x. It follows from the Constant Function Theorem that $f(x) = C$ for some constant C. But then,
$$f(x) = C \Longrightarrow e^{-x}g(x) = C \Longrightarrow g(x) = Ce^x.]$$

23. Let $f(x)$ be a differentiable function such that $f(10) = 1$, $f'(10) = 3$, $f''(10) = 2$ and $f'''(10) = 20$. Approximate $f(11)$.
[Solution. We use Taylor polynomial to approximate $f(x)$ near 10 :

$$\begin{aligned} f(x) &\approx f(10) + f'(10)(x-10) + \frac{f''(10)}{2!}(x-10)^2 + \frac{f'''(10)}{3!}(x-10)^3, \ x \text{ near } 10, \\ &= 1 + 3(x-10) + \frac{2}{2}(x-10)^2 + \frac{20}{6}(x-10)^3 \\ &= 1 + 3(x-10) + 1(x-10)^2 + \frac{10}{3}(x-10)^3. \end{aligned}$$

Therefore,

$$\begin{aligned} f(11) &\approx 1 + 3(11-10) + 1(11-10)^2 + \frac{10}{3}(11-10)^3 \\ &= 1 + 3 + 1 + \frac{10}{3} = \frac{25}{3}.] \end{aligned}$$

End of Midterm Test 1 (Version 11)

SCRAP WORK

SCRAP WORK

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Useful Facts:

Trigonometric Ratios of Special Angles in Quadrant I:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Linearization and linear approximation:

$$f(x) \approx L_a(x) = f(a) + f'(a)(x-a) \text{ for } x \text{ near } a$$

$$\text{Error } E(x) = |f(x) - L_a(x)| \approx \frac{|f''(a)|}{2}(x-a)^2$$

Quadratic approximation:

$$f(x) \approx Q_a(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \text{ for } x \text{ near } a$$

Taylor polynomials and approximations:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$$f(x) = T_n(x) + R, \text{ where } R = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \text{ for some } c \text{ between } a \text{ and } x.$$

(Here, $2! = 2 \times 1,\ 3! = 3 \times 2 \times 1,\ 4! = 4 \times 3 \times 2 \times 1$, etc.)

Recursion Formula for Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

A differential equation with exponential solution:

$$\frac{dy}{dx} = ky \Longrightarrow y = Ce^{kx} \text{ for some constant } C$$