MATH 3705C
Test 4 Solutions

March 24, 2017

[Marks]

[10] 1. The bounded solution of Laplace's equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = 2$ which satisfies the boundary condition $u(2, \theta) = f(\theta)$ has the form

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Find the solution if $f(\theta) = 4 - 2\cos(3\theta) + 3\sin(2\theta)$. Write down the complete solution.

Solution:

$u(2, \theta) = f(\theta) \;\Rightarrow\; \frac{a_0}{2} + \sum_{n=1}^{\infty} 2^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)] = 4 - 2\cos(3\theta) + 3\sin(2\theta) \;\Rightarrow$

$\frac{a_0}{2} = 4$, $2^{-3}a_3 = -2$, $2^{-2}b_2 = 3$, and $a_n = b_n = 0$ otherwise $\Rightarrow$

$a_0 = 8$, $a_3 = -16$, $b_2 = 12$, and $a_n = b_n = 0$ otherwise. Hence,

$u(r, \theta) = 4 - 16r^{-3}\cos(3\theta) + 12r^{-2}\sin(2\theta)$.

[10] 2. The solution of Laplace's equation $u_{xx} + u_{yy} = 0$ within the rectangular region $0 < x < L$, $0 < y < M$, which satisfies the boundary conditions $u(x, 0) = 0$, $u(x, M) = 0$, $u(L, y) = 0$ and $u(0, y) = f(y)$, has the form

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left[\frac{n\pi(L - x)}{M}\right] \sin\left(\frac{n\pi y}{M}\right).$$

Find the solution within the region $0 < x < 2$, $0 < y < 1$, which satisfies the boundary conditions $u(x, 0) = 0$, $u(x, 1) = 0$, $u(2, y) = 0$, and $u(0, y) = y - y^2$. Write down the complete solution $u(x, y)$.

Solution:

$u(0, y) = y - y^2 \;\Rightarrow\; \sum_{n=1}^{\infty} a_n \sinh(2n\pi)\sin(n\pi y) = y - y^2 \;\Rightarrow$

$a_n \sinh(2n\pi) = 2\int_0^1 (y - y^2)\sin(n\pi y)dy$

$= -\frac{2}{n\pi}(y - y^2)\cos(n\pi y)\Big|_0^1 + \frac{2}{n\pi}\int_0^1 (1 - 2y)\cos(n\pi y)dy$

$= \frac{2}{n^2\pi^2}(1 - 2y)\sin(n\pi y)\Big|_0^1 - \frac{2}{n^2\pi^2}\int_0^1 -2\sin(n\pi y)dy$

$= -\frac{4}{n^3\pi^2}\cos(n\pi y)\Big|_0^1 = \frac{4}{n^3\pi^3}[1 - (-1)^n] \;\Rightarrow\; a_n = \frac{4[1 - (-1)^n]}{n^3\pi^3\sinh(2n\pi)} \;\Rightarrow$

$u(x, y) = \sum_{n=1}^{\infty} \frac{4[1 - (-1)^n]}{n^3\pi^3\sinh(2n\pi)} \sinh[n\pi(2 - x)]\sin(n\pi y)$.

[10] 3. The solution of the wave equation $u_{xx} = \frac{1}{c^2} u_{tt}$, $0 < x < L$, $t > 0$, which satisfies the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$, has the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right].$$

Find the solution of $u_{xx} = \frac{1}{4} u_{tt}$, $0 < x < 1$, $t > 0$, which satisfies the boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$, and the initial conditions

$$u(x,0) = 3\sin(\pi x) - \sin(5\pi x) \quad \text{and} \quad u_t(x,0) = \sin(\pi x) - 2\sin(2\pi x).$$

Write down the complete solution $u(x,t)$.

Solution:

$u(x,0) = 3\sin(\pi x) - \sin(5\pi x) \;\Rightarrow\; \sum_{n=1}^{\infty} a_n \sin(n\pi x) = 3\sin(\pi x) - \sin(5\pi x) \;\Rightarrow$
$a_1 = 3$, $a_5 = -1$, and $a_n = 0$ otherwise.

$u_t(x,0) = \sin(\pi x) - 2\sin(2\pi x) \;\Rightarrow\; \sum_{n=1}^{\infty} 2n\pi b_n \sin(n\pi x) = \sin(\pi x) - 2\sin(2\pi x) \;\Rightarrow$
$2\pi b_1 = 1$, $4\pi b_2 = -2$, and $b_n = 0$ otherwise $\Rightarrow$ $b_1 = \frac{1}{2\pi}$, $b_2 = \frac{1}{2\pi}$, and $b_n = 0$ otherwise. Hence, $u(x,t) =$
$3\sin(\pi x)\cos(2\pi t) - \sin(5\pi x)\cos(10\pi t) + \frac{1}{2\pi}\sin(\pi x)\sin(2\pi t) - \frac{1}{2\pi}\sin(2\pi x)\sin(4\pi t).$