MATH 3705C
Test 3 Solutions
March 10, 2017

[3] 1. Let $f(x) = x^3$ on $[0, 3]$. At $x = 22$, the Fourier series of f converges to
(a) 0 (b) 1 (c) 8 (d) -8 (e) None of these
Answer: (b)

[3] 2. Let $f(x) = x^3$ on $[0, 3]$. At $x = 22$, the Fourier sine series of f converges to
(a) 0 (b) 1 (c) 8 (d) -8 (e) None of these
Answer: (d)

[3] 3. Let $f(x) = x^3$ on $[0, 3]$. At $x = 22$, the Fourier cosine series of f converges to
(a) 0 (b) 1 (c) 8 (d) -8 (e) None of these
Answer: (c)

[7] 4. The solution of the heat equation $u_{xx} = \frac{1}{\alpha^2} u_t$, $0 < x < L$, $t > 0$, which satisfies the boundary conditions $u(0, t) = u(L, t) = 0$, $t > 0$, has the form

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

Find the solution $u(x, t)$ of $u_{xx} = \frac{1}{4} u_t$, $0 < x < 1$, $t > 0$, which satisfies the boundary conditions $u(0, t) = u(1, t) = 0$ and the initial condition $u(x, 0) = 1$. Write down the complete solution $u(x, t)$.

Solution:

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = 1 \;\Rightarrow\; b_n = 2\int_0^1 \sin(n\pi x)dx = -\frac{2}{n\pi}\cos(n\pi x)\Big|_0^1$$

$$= \frac{2[1 - (-1)^n]}{n\pi} \;\Rightarrow\; u(x, t) = \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n\pi} \sin(n\pi x) e^{-4n^2\pi^2 t}.$$

[7] 5. Find the solution $u(x, t)$ of $u_{xx} = \frac{1}{4} u_t$, $0 < x < 1$, $t > 0$, which satisfies the boundary conditions $u(0, t) = 2$, $u(1, t) = 3$, and the initial condition $u(x, 0) = x + 3$. Write down the complete solution $u(x, t)$. *Hint:* $u(x, t) = v(x) + w(x, t)$.

Solution:

$v''(x) = 0 \;\Rightarrow\; v(x) = ax + b$, and $v(0) = 2$ and $v(1) = 3 \;\Rightarrow\; v(x) = x + 2$.

$w_{xx} = \frac{1}{4} w_t$ and $w(0, t) = w(1, t) = 0 \;\Rightarrow\; w(x, t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-4n^2\pi^2 t}$. Then

$w(x, 0) = u(x, 0) - v(x) = 1 \;\Rightarrow\; b_n = \frac{2[1 - (-1)^n]}{n\pi}$, as in Problem 4. Hence,

$$u(x, t) = x + 2 + \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n\pi} \sin(n\pi x) e^{-4n^2\pi^2 t}.$$

[7] 6. The solution of the heat equation $u_{xx} = \frac{1}{\alpha^2} u_t,\ 0 < x < L,\ t > 0$, which satisfies the boundary conditions $u_x(0,t) = u_x(L,t) = 0,\ t > 0$, has the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

Find the solution $u(x,t)$ of $u_{xx} = \frac{1}{9} u_t,\ 0 < x < 1,\ t > 0$, which satisfies the boundary conditions $u_x(0,t) = u_x(1,t) = 0$ and the initial condition $u(x,0) = 2x$. Write down the complete solution $u(x,t)$.

Solution:

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = 2x \ \Rightarrow$$

$$a_n = 2\int_0^1 2x \cos(n\pi x)\, dx = \frac{4}{n\pi} x \sin(n\pi x)\Big|_0^1 - \frac{4}{n\pi}\int_0^1 \sin(n\pi x)\, dx = \frac{4}{n^2\pi^2}\cos(n\pi x)\Big|_0^1$$

$= \frac{4[(-1)^n - 1]}{n^2\pi^2}$, $n \geq 1$, and $a_0 = 2\int_0^1 2x\, dx = 2$. Hence,

$$u(x,t) = 1 + \sum_{n=1}^{\infty} \frac{4[(-1)^n - 1]}{n^2\pi^2} \cos(n\pi x) e^{-9n^2\pi^2 t}.$$