MATH 3705C
Test 2 Solutions
February 17, 2017

[4] 1. For the equation $x^2y'' - y' + \frac{1}{(x-1)^2}y = 0$,
(a) All points are ordinary.
(b) 0 is irregular singular and 1 is regular singular.
(c) 0 is regular singular and 1 is irregular singular.
(d) Both 0 and 1 are regular singular.
(e) Both 0 and 1 are irregular singular.
Answer: (b)

[4] 2. The general solution of the equation $x^2y'' + 3xy' + 5y = 0$ for $x \neq 0$ is $y =$
(a) $|x|^{-3/2}[c_1 \cos(\frac{\sqrt{11}}{2} \ln|x|) + c_2 \sin(\frac{\sqrt{11}}{2} \ln|x|)]$ (b) $x^{-1}[c_1 \cos(\ln|x|) + c_2 \sin(\ln|x|)]$
(c) $x^{-1}[c_1 + c_2 \ln|x|]$ (d) $e^{-x}[c_1 \cos(x) + c_2 \sin(x)]$ (e) None of the these
Answer: (e)

[4] 3. The general solution of the equation $x^2y'' + 3xy' + y = 0$ for $x \neq 0$ is $y =$
(a) $|x|^{-3/2}[c_1 \cos(\frac{\sqrt{5}}{2} \ln|x|) + c_2 \sin(\frac{\sqrt{5}}{2} \ln|x|)]$ (b) $x^{-1}[c_1 + c_2 \ln|x|]$ (c) $c_1x^{-1} + c_2x^{-1}$
(d) $c_1|x|^{(-3+\sqrt{5})/2} + c_2|x|^{(-3-\sqrt{5})/2}$ (e) None of the these
Answer: (b)

[4] 4. The general solution of the equation $x^2y'' - 6y = 0$ for $x \neq 0$ is $y =$
(a) $c_1x^2 + c_2x^{-3}$ (b) $c_1|x|^{\sqrt{6}} + c_2|x|^{-\sqrt{6}}$ (c) $c_1J_0(\sqrt{6}x) + c_2Y_0(\sqrt{6}x)$
(d) $c_1x^3 + c_2x^{-2}$ (e) None of these
Answer: (d)

[4] 5. The general solution of the equation $x^2y'' + xy' + (5x^2 - 4)y = 0$ for $x > 0$ is $y =$
(a) $c_1J_2(\sqrt{5}x) + c_2Y_2(\sqrt{5}x)$ (b) $c_1J_2(\sqrt{5}x) + c_2J_{-2}(\sqrt{5}x)$ (c) $c_1J_{\sqrt{5}}(2x) + c_2J_{-\sqrt{5}}(2x)$
(d) $c_1J_{\sqrt{5}}(2x) + c_2Y_{\sqrt{5}}(2x)$ (e) None of these
Answer: (a)

[10] 6. Find one (nonzero) solution y_1 of $x^2y'' - xy' + (1+x)y = 0$ about $x_0 = 0$ for $x > 0$.
Solution:
$p(x) = -\frac{1}{x}$ and $q(x) = \frac{1+x}{x^2}$ are singular at 0, and $xp(x) = -1$ and $x^2q(x) = 1+x$ are analytic at 0. Hence, 0 is a regular singular point, with $p_0 = -1$ and $q_0 = 1$. The indicial equation is $r^2 + (p_0 - 1)r + q_0 = r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r = 1$. Hence, one solution takes the form
$y_1 = \sum_{n=0}^{\infty} a_n x^{n+1}$. Then $y_1' = \sum_{n=0}^{\infty}(n+1)a_n x^n$, $y_1'' = \sum_{n=0}^{\infty} n(n+1)a_n x^{n-1}$,
and the equation requires that

$$\sum_{n=0}^{\infty} n(n+1)a_n x^{n+1} - \sum_{n=0}^{\infty}(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \Rightarrow$$

$$\sum_{n=0}^{\infty} n^2 a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0 \Rightarrow \sum_{n=0}^{\infty}\left[(n+1)^2 a_{n+1} + a_n\right] x^{n+2} = 0 \Rightarrow$$

$a_{n+1} = \frac{-a_n}{(n+1)^2}$, $n \geq 0$. Then

$n = 0 \;\Rightarrow\; a_1 = \dfrac{-a_0}{1^2}$,

$n = 1 \;\Rightarrow\; a_2 = \dfrac{-a_1}{2^2} = \dfrac{a_0}{1^2 \cdot 2^2}$,

$n = 2 \;\Rightarrow\; a_3 = \dfrac{-a_2}{3^2} = \dfrac{-a_0}{1^2 \cdot 2^2 \cdot 3^2}$, etc., and $a_n = \dfrac{(-1)^n a_0}{(n!)^2}$, $n \geq 0$.

Thus, $y_1 = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^{n+1}$.