University of Ottawa

.MAT 2377 – Midterm I, Winter 2018.

Feb 15, 2018 **Professor: Chen Xu and Termeh Kousha**

Time: 80 minutes

Student Number:_________

Family Name : __________ **First Name:** ________________________

This is a closed book examination. Only basic calculators are allowed. Record your answer to each of the multiple choice questions in the table below (NOT for Question 7 and 8).

Grade:_________**/15**

Question	Answer	Your Grade
1	B	
2	B	
3	B	
4	A	
5	C	
6	E	
7		
8		

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You MUST sign below
Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Multiple Choice Questions (1 marks/question for a total 6 marks)

Please Record your answer to each multiple choice question in the table on the first page.

1. If the probability density function of a random variable X is

$$f_X(x) = k\,x^3, \quad 0 < x < 1.$$

Compute the probability that X will be between 1/4 and 3/4.
(Hint: first you need to find the value of k)

A) 0.0781 B) 0.3125 C) 0.625 D) 0.4523 E) 0.9267
The answer is B.
Solution:

$$1 = \int_0^1 k\,x^3\,dx = \frac{k}{4} \quad \Rightarrow \quad k = 4$$

$$P(1/4 < X < 3/4) = \int_{1/4}^{3/4} 4\,x^3\,dx = (3/4)^4 - (1/4)^4 = 5/16 = 0.3125$$

2. Suppose there are 14 boys and 6 girls in a classroom. A teacher randomly selects 5 students from the class without replacement. What is the probably that the selected students contain 2 girls?

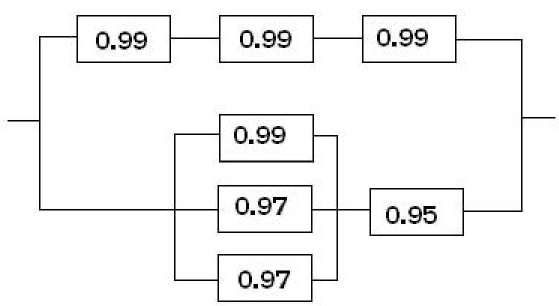
   A) 0.079 B) 0.352 C) 0.001 D) 0.024 E) 0.309

   The answer is B. **Solution:**

   Let A be the event that the selected students contain 2 girls (and 3 boys). Note that order of selection is not important, we have

$$P(A) = \frac{n(A)}{n(S)} = \frac{{}_6C_2 \times {}_{14}C_3}{{}_{20}C_5} = 0.352$$

3. The following circuit is operational only if there is a path of functional devices from left to right. What is the probability that the circuit operational?



   A) 0.9744 B) 0.9985 C) 0.9898 D) 0.7566 E) 0.0222
   The answer is B.
   Solution: The probability that the top circuit in series works is $(0.99)^3 = 0.970299$. In the bottom part of the circuit, we have a circuit of 3 components in parallel. The probability that the latter sub-circuit works is $1 - (.01)(.03)(.03) = 0.999991$. This latter sub-circuit is put in series with another component. The probability that this bottom sub-circuit works is $(0.999991)(0.95) = 0.94999145$. The circuit is composed of

the top sub-circuit and the bottom sub-circuit that are put together in parallel. The probability that the circuit works is

$$1 - (1 - 0.970299)(1 - 0.94999145) = 0.9985.$$

4. Suppose that Company I and Company II are two manufacturers of a certain microchip. In the market, Company I makes 60% of this type of microchips and Company II makes the rest 40%. It is known that, 5% of the Company I microchips are detective, while 3% of the Company II microchips are detective. Suppose we randomly select a microchip from the market and find that it is not detective, what is the probably that this microchip is made by Company II?

   A) 0.405 B) 0.400 C) 0.286 D) 0.595 E) 0.388

   The answer is A.
   Solution:
   Let A denote the event that a randomly selected microchip is made by Company I. Let E denote the event that a randomly selected microchip is detective. From the question, we know that

$$P(A) = 60\%, \ P(A^c) = 40\%, \ P(E|A) = 5\%, \ P(E|A^c) = 3\%$$

   Thus,

$$P(E) = P(A)P(E|A) + P(A^c)P(E|A^c) = 0.6 \times 0.05 + 0.4 \times 0.03 = 0.042$$

   The question asks us to compute

$$P(A^c|E^c) = \frac{P(A^c \cap E^c)}{P(E^c)} = \frac{P(A^c)P(E^c|A^c)}{1 - P(E)} = \frac{0.4 * 0.97}{1 - 0.042} = 0.405$$

5. An Athlete suspected of having used steroids is given two tests that operate independently of each other if steroids have been used. Test A has a probability of 0.9 of being positive if steroids have been used.

Test B has probability of 0.8 of being positive if steroids have been used. What is the probability that *neither* test is positive if steroids have been used.

A) 0.72 B) 0.38 C) 0.02 D) 0.06 E) 0.32

The answer is C.
Solution:
Let us assume that steroids have been used. Let A be the event that Test A is positive and let B the event that Test B is positive. Since A and B are independent, then $P(A \cap B) = P(A)P(B) = (0.9)(0.8) = 0.72$. The probability that *neither* test is positive if steroids have been used is:

$$\begin{aligned} P(A' \cap B') &= 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - (0.9 + 0.8 - 0.72) = 0.02. \end{aligned}$$

6. Let X denote the number of students who show up during the office hours of MAT2377 in a certain week. Let $F(x)$ denote the *cdf* of X and $f(x)$ denote the *pmf* of X. Suppose X takes values from $\{0, 1, 2, 3, 4\}$ and its *pmf* is given as follows.

x	0	1	2	3	4
$f(x)$	0.23	0.34	0.17	0.15	?

Which of the following statements is **wrong**?
A) $f(4) = 0.11$
B) $E(X) = 1.57$
C) $F(0) = f(0)$
D) $F(2) > f(2)$

E) $F(3.5) = 0$

The answer is E.
Solution: By definition, $F(3.5) = P(X \leq 3.5) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \neq 0$.

Short answer questions (total of 9 marks)

7. (6 points) Suppose that the joint pmf of the random variables X and Y is given by the following table:

x	y	$f_{XY}(x,y)$
1	1	1/4
1.5	2	1/8
1.5	3	1/4
2.5	4	1/4
3	5	1/8

Determine the following probabilities:
(a) $P(X < 2.5, Y < 3)$

(b) $P(Y > 2.8 | X = 1.5)$

(c) If $E[Y] = 2.875$, Compute $E(2X - 3Y)$

(d) If the $VAR(X) = 0.496$ and $VAR(Y) = 1.859$, compute the correlation between X and Y.

(e) Are X and Y independent? Why?
Solution:
(a) $P(X < 2.5, Y < 3) = P(X = 1, Y = 1) + P(X = 1.5, Y = 2) = \frac{3}{8}$

(b) $P(Y > 2.8 | X = 1.5) = \frac{P(Y>2.8, X=1.5)}{P(X=1.5)} = \frac{P(X=1.5,Y=3)}{P(X=1.5,Y=2)+P(X=1.5,Y=3)} = \frac{\frac{1}{4}}{\frac{3}{8}} = \frac{2}{3}$

(c)

$$E(X) = (1)(\frac{1}{4}) + (1.5)(\frac{3}{8}) + (2.5)(\frac{1}{4}) + (3)(\frac{1}{8}) = 1.8125$$

$$E(Y) = (1)(\frac{1}{4}) + (2)(\frac{1}{8}) + (3)(\frac{1}{4}) + (4)(\frac{1}{4}) + (5)(\frac{1}{8}) = 2.875$$

So

$$E(2X - 3Y) = 2E(X) - 3E(Y) = 2(1.8125) - 3(2.875) = -5$$

(d)

$$E(X,Y) = (1)(1)(\frac{1}{4}) + (1.5)(2)(\frac{1}{8}) + (1.5)(3)(\frac{1}{4}) + (2.5)(4)(\frac{1}{4}) + (3)(5)(\frac{1}{8}) = 6.125$$

$$COV(X,Y) = E(XY) - E(X)E(Y) = 6.125 - (2.875)(1.8125) = 0.914$$

$$\rho = \frac{0.914}{\sqrt{0.496}\sqrt{1.859}} = 0.69811$$

(e) No since $COV(X,Y) \neq 0$.

Marking Scheme: For (a), (b) and (e) 0.5 each. 3 points for (c) : 1 for $E(X)$ 1 for $E(Y)$ 1 for $E(2X+3Y)$. 1.5 point for (d):1 for Covariance and 0.5 for variance. 6 points total.

8. (3 points) Suppose that 2 % of the transistors from a production line are defective. It is believed that the quality of those transistors are independent. Every hour, we take a random sample of 10 transistors from the production line. If a sample contains more than 1 defective unites, we must interrupt the production line.

   (a) What is the probably that we need to interrupt the production line after the 1st hour?

   (b) On average, how many hours are required to observe the 1st interruption?

   (c) Compute the standard deviation of the number of hours required to observe 3 interruptions of the production line.

   Solution:

   a) Let X denote the number of defective unites in a random sample of 10. From the question, we know that $X \sim B(n = 10, p = 0.02)$, where "success" means a defective unit. Thus, the question asks us to compute

$$\begin{aligned} P(X > 1) &= 1 - P(X \leq 1) = 1 - P(X = 0) - P(X = 1) \\ &= 1 - 0.98^{10} - 10 \times 0.98^{9} 0.02 = 0.016 \end{aligned}$$

   b) Let Y denote the number of hours required to observe r interruptions. We can see that $Y \sim NB(r, q)$ with $q = 0.016$ being the success rate, where "success" means the interruption of production line. For part (b), $r = 1$ and thus $Y \sim Geo(q)$.

$$E(Y) = 1/q = 1/0.016 = 62.5 \text{ (hours)}.$$

   c) For part c), $Y \sim NB(r, q)$ with $r = 3$ and $q = 0.016$.

$$Sd(Y) = \sqrt{Var(Y)} = \sqrt{\frac{r(1-q)}{q^2}} = \sqrt{\frac{3(1-0.016)}{0.016^2}} = 107.384$$

   Marking Scheme: 1 point each part

(Question 8 cont.)

Extra page for calculations

Extra page for calculations