Solutions to the final examination for MATH 1009* ABCD, Winter 2014.

PART A. MULTIPLE CHOICE QUESTIONS.

Answers: dacbc babda daccb dad

1. The domain of the function $f(x) = \dfrac{1}{\sqrt{(x-1)(x-2)}}$ is

a) $[1,2]$, or equivalently, $\{x : 1 \le x \le 2\}$.

b) $(1,2)$, or equivalently, $\{x : 1 < x < 2\}$.

c) $(-\infty, 1] \bigcup [2, +\infty)$, or equivalently, $\{x : x \le 1 \text{ and } x \ge 2\}$.

d) $(-\infty, 1) \bigcup (2, +\infty)$, or equivalently, $\{x : x < 1 \text{ and } x > 2\}$.

e) None of the above.

Answer: d)

2. Let $f(x) = 2x^3 - 1$ and $g(x) = \dfrac{1}{x}$. Then the value of the composition $f(g(1))$ is

a) 1. b) $-\dfrac{1}{3}$. c) -1. d) -3. e) None of the above.

Answer: a)

3. If $e^{x+2} = 3$, what is x ?

a) $\ln 2$. b) $3 - \ln 2$. c) $\ln 3 - 2$. d) $\dfrac{\ln 3}{2}$. e) None of the above.

Answer : c)

4. The expression $\dfrac{(x^{0.4})^2 \cdot x^{-3.8}}{x^2}$ simplifies to

a) x^5. b) x^{-5}. c) x. d) x^{-1}. (e) None of the above.

Answer: (b)

5. Which of the following is equal to $\log_{\frac{1}{2}} \dfrac{1}{8}$?

a) $\dfrac{1}{3}$. b) $-\dfrac{1}{3}$. c) 3. d) -3. e) None of the above.

Answer: c)

6. The statement $3\ln(2x) - \ln(x^2) + \ln 5$ written as a single logarithm is

a) $\ln x$. b) $\ln(40x)$. c) $\ln(8x^3 - x^2 + 5)$. d) $\ln(5 - x^2)$. e) None of the above.

Answer: b)

7. What is $\lim_{x\to\infty} \dfrac{4x^3 + x^2 - 3}{1 + 2x - x^3}$?

a) -4. b) -3. c) 0. d) ∞. e) None of the above.

Answer: a)

8. What is the slope of the curve $y = 3x^{2/3}$ at $x = \dfrac{1}{8}$?

a) 1. b) 4. c) $\dfrac{1}{4}$. d) $\dfrac{4}{3}$. e) None of the above.

Answer: b)

9. Given that the profit function for a company is

$$P(x) = x^2 + \frac{50}{x} + 10x - 7,$$

find the **marginal profit** at a production level of $x = 5$ units.

a) 78. b) 87. c) 22. d) 18. e) None of the above.

Answer: d)

10. The derivative f' of $f(x) = (3x^2 - e^x)^7$ is

a) $7(3x^2 - e^x)^6 \cdot (6x - e^x)$. b) $7(3x^2 - e^x)^6 \cdot (6x - 1)$. c) $(3x^2 - e^x)^6 \cdot (6x - e^x)$.

d) $7(3x^2 - e^x) \cdot (6x - 1)$. e) None of the above.

Answer: a)

11. What are the critical numbers of the function $f(x) = \dfrac{1}{x+2}$?

a) -2. b) 0. c) 2. d) No critical numbers. e) None of the above.

Answer: d)

12. The second derivative f'' of $f(x) = \ln(3x-1)$, $(x > \frac{1}{3})$ is

a) $\dfrac{-9}{(3x-1)^2}$. b) $\dfrac{3}{(3x-1)^2}$. c) $\dfrac{-9x}{3x-1}$. d) $\dfrac{3}{3x-1}$. e) None of the above.

Answer: a)

13. The graph of the function $y = \dfrac{x^2}{x^2+1}$ has

a) no horizontal asymptote and a vertical asymptote $x = -1$.

b) a horizontal asymptote $y = 1$ and a vertical asymptote $x = -1$.

c) a horizontal asymptote $y = 1$ and no vertical asymptote.

d) neither horizontal no vertical asymptote.

e) none of the above.

Answer: c)

14. Consider the function $f(x) = x^3 - 3x^2 + 1$. The function is increasing when:

a) $x \in (0, 2)$.

b) $x \in (0, \infty)$.

c) $x \in (-\infty, 0) \bigcup (2, \infty)$.

d) $x \in \mathbb{R}$.

e) None of the above.

Answer: c)

15. Let $f(x,y) = \frac{x^4}{6} - x^2 + \frac{5}{6}$. What are the inflection points of the function (if any)?

a) $(0,-1)$ and $(0,1)$. b) $(-1,0)$ and $(1,0)$. c) $(1,0)$.

d) No inflection points. e) None of the above.

Answer: b)

16. Let $f(x,y) = e^{x+3y}$. What is $f_{yy}(1,0)$?

a) 1. b) e. c) $3e$. d) $9e$. e) None of the above.

Answer: d)

17. Consider the Cobb-Douglas production function $f(x,y) = 9x^{2/3}y^{1/3}$, where x is the number of units of labour and y is the number of units of capital. What is the marginal productivity of **labour** at $x = 27$, $y = 64$?

a) 8. b) 16. c) 256. d) 432. e) None of the above.

Answer: a)

18. The value of $\int_{-1}^{2} (x+1)\,dx$ is

a) -2.5. b) -1. c) 3. d) 4.5. e) None of the above.

Answer: d)

PART B. Answer all questions and show all appropriate steps in your work; otherwise only partial marks may be awarded.

B1. [**9 Marks**] An amount of \$10,000 is deposited in a bank that pays interest at the rate of 6% per year. Answer the questions below, rounding the answers to two decimal points.

[2] **(a)** If the money is invested for the period of 5 year compounded **quarterly**, what would be the **interest** earned?

[3] **(b)** How long would it take for the investment to grow to \$12,000 if the interest is compounded **semiannually**?

[1] **(c)** What would be the accumulated amount (or return) after 10 years from the investment if the interest is compounded **continuously**?

[3] **(d)** What should the interest rate be with **continuous compounding** in order to double the original amount of money after 10 years?

Solution:

(a) $A(t) = P\left(1 + \frac{r}{m}\right)^{mt} = 10,000\left(1 + \frac{0.06}{4}\right)^{4 \cdot 5} = 10,000(1.015)^{20} = 13,468.55;$

$I = A - P = 3,468.55.$

(b) $A(t) = P\left(1 + \frac{r}{m}\right)^{mt} \quad \Rightarrow \; 12,000 = 10,000\left(1 + \frac{0.06}{2}\right)^{2 \cdot t} \quad \Rightarrow \; 1.2 = 1.03^{2t}$

$\Rightarrow \; \ln(1.2) = 2t\ln(1.03) \quad \Rightarrow t = \frac{\ln(1.2)}{2\ln(1.03)} \approx 3.08.$

(c)

$$A(t) = Pe^{rt} = 10,000\, e^{0.06 \cdot 10} = 18,221.19.$$

(d)

$$A(t) = 2P = Pe^{r \cdot 10}; \text{ Solve for } r: \; e^{10r} = 2, \; 10r = \ln 2, \; r = \frac{\ln 2}{10} = 0.0693 = 6.93\%.$$

B2. [12 Marks]

Use the method of Lagrange multipliers to find the maximum and the minimum values of the function

$$f(x, y) = xy$$

subject to the constraint

$$x^2 + y^2 = 8.$$

Solution:

$$F(x, y, \lambda) = xy + \lambda(x^2 + y^2 - 8).$$

$$\begin{cases} F_x = y & + & 2\mathrm{x}\lambda = 0, & (1) \\ F_y = \mathrm{x} & + & 2\mathrm{y}\lambda = 0, & (2) \\ F_\lambda = x^2 & + & y^2 - 8 = 0. & (3) \end{cases}$$

From (1) $\Rightarrow$ $\lambda = -\dfrac{y}{2x}$, from (2) $\Rightarrow$ $\lambda = -\dfrac{x}{2y}$.

Equate λ to obtain $-\dfrac{y}{2x} = -\dfrac{x}{2y}$, or $x^2 = y^2$.

Substituting $y^2 = x^2$ into (3) yields

$$x^2 + x^2 - 8 = 0 \;\Rightarrow\; 2x^2 = 8 \;\Rightarrow\; x^2 = 4 \Rightarrow\; x = \pm 2$$

.

Thus, $y^2 = x^2 = 4$, $y = \pm 2$.

The critical points are $(-2, -2)$, $(-2, 2)$, $(2, -2)$, $(2, 2)$. Evaluate $f(x, y)$ at each of the critical points to find the minimum and the maximum value:

$f(-2, -2) = f(2, 2) = 4$ is the maximum value; $f(2, -2) = f(-2, 2) = -4$ is the minimum value.

B3. [12 Marks]

[5] **(a)** Find all the critical points of the function $f(x, y) = x^2 - 4xy + 2y^2 + 4x + 8y - 1$. (Do NOT classify them).

Solution:

$f_x = 2x - 4y + 4 = 0;$

$f_y = -4x + 4y + 8 = 0;$

Add the equations to get $-2x + 12 = 0$ or $x = 6$. Then from the first equation

$$4y = 2x + 4, \quad y = \frac{x}{2} + 1 = \frac{6}{2} + 1 = 4$$

.

Thus, the critical point is $(6, 4)$.

[7] **(b)** The function $f(x, y) = 2x^3 - 2xy + y^2 + 7$ has two critical points $(0, 0)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$. Use the Second Derivative Test to classify the nature of each point, if possible.

The Second Derivative Test:

$f_x = 6x^2 - 2y;$

$f_y = -2x + 2y;$

$f_{xx} = 12x, \quad f_{yy} = 2, \quad f_{xy} = -2,$ so $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 24x - 4.$

$D(0, 0) = -4 < 0,$ therefore $(0, 0)$ is a saddle point (neither a relative min nor a relative max).

$D\left(\frac{1}{3}, \frac{1}{3}\right) = 24 \cdot \left(\frac{1}{3}\right) - 4 = 4 > 0, \quad f_{xx} = 4 > 0,$ therefore this is a local (relative) minimum point.

B4. [13 Marks]

(a) Evaluate the following integrals:

[2] (i) $x > 0,\quad \int \left(2x^5 + \frac{1}{x}\right) dx = 2\int x^5\,dx + \int \frac{1}{x}\,dx = 2\cdot\frac{x^6}{6} + \ln x + C = \frac{x^6}{3} + \ln x + C.$

[4] (ii) $\int_0^2 (3x-2)^3\,dx$

Let $u = 3x - 2$, then $\frac{du}{dx} = 3,\ du = 3\,dx,\ dx = \frac{du}{3}.$

Change the limits of integration: $x = 0 \Rightarrow u = 3x - 2 = -2, \quad x = 2 \Rightarrow u = 3x - 2 = 4.$ Thus,

$$\int_0^2 (3x-2)^3\,dx = \int_{-2}^4 u^3\,\frac{du}{3} = \frac{1}{3}\int_{-2}^4 u^3\,du = \frac{1}{3}\cdot\frac{u^4}{4}\Big|_{-2}^4 = \frac{1}{12}[4^4 - (-2)^4] = \frac{240}{12} = 20.$$

[3] (iii) $\int 2e^{1-6x}\,dx = 2\int e^u \cdot \left(\frac{du}{-6}\right) = -\frac{1}{3}\int e^u\,du = -\frac{1}{3}\,e^u + C = -\frac{1}{3}\,e^{1-6x} + C$

[4] **(b)** Consider the equation $x^3 - 2xy + y^4 = 5$, where $y = y(x)$ is defined implicitly as a function of x. Find $\frac{dy}{dx}$.

Rewrite expression, stating that $y = y(x)$ is a function of x:

$x^3 - 2x\cdot y(x) + (y(x))^4 = 5.$

Differentiate both parts of the above expression with respect to x:

$3x^2 - 2\cdot y(x) - 2x\cdot y'(x) + 4(y(x))^3\cdot y'(x) = 0.$

Solve for $y'(x)$:

$-2x\cdot y'(x) + 4y^3\cdot y'(x) = -3x^2 + 2y, \qquad y'(x)(4y^3 - 2x) = 2y - 3x^2, \quad y'(x) = \frac{2y - 3x^2}{4y^3 - 2x}.$