Name ____

Student number ______________________________________

Lecture section (L1 - Dr. V. Dimitrov; L2 - Dr. Y. Wang)______________

University of Calgary
Schulich School of Engineering
Fall 2011 Final Examination

ENGG 407
Numerical Methods

December 15, 2011, 12:00-3:00pm

3 Hours Duration

1. Examination is closed book.
2. No calculators
3. You do not need to simplify the numerical expressions unless stated
4. All angles are in radians, for example sin(x), x is assumed to be in radians
5. Final exam counts for 40% of overall course grade.
6. Exam has 20 multiple choice questions. Write the answers in the spaces provided in this exam booklet.
7. Exam has 5 written questions. Write answers in the space provided below each question.
8. Total marks for the exam is 100. Marks value of each question is indicated.

Question	Area	Value	Mark
1	Multiple choice questions	20	
2	Curve fitting, interpolation, and integration	20	
3	Numerical differentiation and integration	15	
4	ODE with initial values and boundary values	20	
5	ODE and differentiations	15	
6	Applied numerical integration in engineering	10	
Total		**100**	

EXAMINATION RULES AND REGULATIONS

STUDENT IDENTIFICATION

Each candidate must sign the Seating List confirming presence at the examination. All candidates for final examinations are required to place their University of Calgary I.D. cards on their desks for the duration of the examination. (Students writing mid-term tests can also be asked to provide identity proof.) Students without an I.D. card who can produce an acceptable alternative I.D., e.g., one with a printed name and photograph, are allowed to write the examination.

A student without acceptable I.D. will be required to complete an Identification Form. The form indicates that there is no guarantee that the examination paper will be graded if any discrepancies in identification are discovered after verification with the student's file. A Student who refuses to produce identification or who refuses to complete and sign the Identification Form is not permitted to write the examination.

EXAMINATION RULES

(1) Students late in arriving will not normally be admitted after one-half hour of the examination time has passed.

(2) No candidate will be permitted to leave the examination room until one-half hour has elapsed after the opening of the examination, nor during the last 15 minutes of the examination. All candidates remaining during the last 15 minutes of the examination period must remain at their desks until their papers have been collected by an invigilator.

(3) All inquiries and requests must be addressed to supervisors only.

(4) Candidates are strictly cautioned against:

(a) speaking to other candidates or communicating with them under any circumstances whatsoever;

(b) bringing into the examination room any textbook, notebook or memoranda not authorized by the examiner;

(c) making use of calculators and/or portable computing machines not authorized by the instructor;

(d) leaving answer papers exposed to view;

(e) attempting to read other student's examination papers.

The penalty for violation of these rules is suspension or expulsion or such other penalty as may be determined.

(5) Candidates are requested to write on both sides of the page, unless the examiner has asked that the left hand page be reserved for rough drafts or calculations.

(6) Discarded matter is to be struck out and not removed by mutilation of the examination answer book.

(7) Candidates are cautioned against writing in their answer book any matter extraneous to the actual answering of the question set.

(8) The candidate is to write his/her name on each answer book as directed and is to number each book.

(9) A candidate must report to a supervisor before leaving the examination room.

(10) Answer books must be handed to the supervisor-in-charge promptly when the signal is given. Failure to comply with this regulation will be cause for rejection of an answer paper.

(11) If during the course of an examination a student becomes ill or receives word of a domestic affliction, the student should report at once to the supervisor, hand in the unfinished paper and request that it be cancelled. If physical and/or emotional ill health is the cause, the student must report at once to a physician/counsellor so that subsequent application for a deferred examination is supported by a completed Physician/Counsellor Statement form. Students can consult professionals at University Health Services or University Counselling Services during normal working hours or consult their physician/counsellor in the community.

Should a student write an examination, hand in the paper for marking, and later report extenuating circumstances to support a request for cancellation of the paper and for another examination, such a request will be denied.

(12) Smoking during examinations is strictly prohibited.

RO 94-01

1. Multiple choice questions (20, each question valued at one mark)

[d] 1. What is/are the characteristics of numerical methods that distinguish them from analytic mathematics?
a. Using approximation techniques to find solutions for an analytic function
b. Using iterative and/or recursive algorithms
c. Suitable for problems that cannot be or difficult to be solved analytically
d. All the above

[a] 2. Across all fields of numerical methods, which method you'd think as the most fundamental and important method where more other methods are based on?
a. Taylor (Taylor series)
b. Newton (Derivative)
c. Gauss (Gauss quadrature integration)
d. Euler (Euler's methods for ODEs)

[c] 3. Which of the following is *NOT* a typical method for numerical solutions for ODEs with initial values (IVs)?
a. Euler's methods
b. The midpoint method
c. Gaussian method
d. The predictor-corrector methods

[c] 4. Taylor series may be applied in the following numerical methods *except*:
a. Numerical differentiation
b. Function expansion
c. Numerical integration
d. Truncation error estimation

[a] 5. Given $\frac{dy}{dx} = x - y$, with IV $y(0) = 2$, $0 \le x \le 1$, $h = 0.5$, determine *y(x | x=0.5)* by *Euler's explicit* methods.

a. 1 b. 2 c. 3.75 d. 4.75

[b] 6. Which of the following is *NOT* a typical method for numerical optimization?
a. Newton's method
b. The searching for df(x)/dx = 0
c. The binary section search method
d. The random search method

[d] 7. Given $\frac{dy}{dx} = x - y$, with IV $y(0) = 2$, $0 \le x \le 1$, $h = 0.5$, determine *y(x | x=0.5)* by the *predictor-corrector* method.

a. 3.75 b. 2 c. 3 d. 1.375

[b] 8. Which of the following is *NOT* a typical method for numerical solutions of nonlinear equations?
a. The Bisection method
b. The spline method
c. The Regula-Falsi method
d. Newton's method

[b] 9. Which of the following methods does not perform well in dealing with stiff ODEs?
a. Runge Kutta methods
b. Euler's explicit method
c. Predictor-corrector methods
d. Euler's implicit method

[a] 10. What is the expected function that satisfies $\frac{d^2y}{dx^2} = \frac{dy}{dx} = y(x)$?

a. e^x
b. x^e
c. $y = x^2$
d. $y = x$

[c] 11. Which of the following is *NOT* a typical method for numerical solutions of linear equations?
a. The Gaussian elimination method
b. Jacobi's method
c. The pseudo inverse method
d. The Gauss-Seidel method

[d] 12. The number of potential solution(s) for a first order ODE without the initial value specified is:
a. 0
b. 1
c. uncertain
d. infinite

[a] 13. Which of the following is *NOT* a typical method for curve fitting and interpolation?
a. The fix-point regression method
b. The Polynomial regression method
c. The linear regression method
d. The spline method

[b] 14. In order to uniquely solve a system of *n* second-order ODEs, how many initial values are needed?
a. n+1
b. 2n
c. n^2
d. n

[c] 15. Which of the following is *NOT* a typical method for numerical integration?
a. The finite integration method
b. Simpson's (polynomial) method
c. The Taylor series method
d. The Gauss quadrature method

[c] 16. What is a more accurate numerical integration result using the Richardson's extrapolation method given $I_{\text{midpoint}}(h_1 \mid O(h_1^2)) = 99,$ and $I_{\text{Trapezoidal}}(h_2 \mid O(h_2^2)) = 102$ where $h = h_2 = \frac{1}{2}h_1 = 0.5$?
a. 101
b. 100.5
c. 103
d. Undeterminable

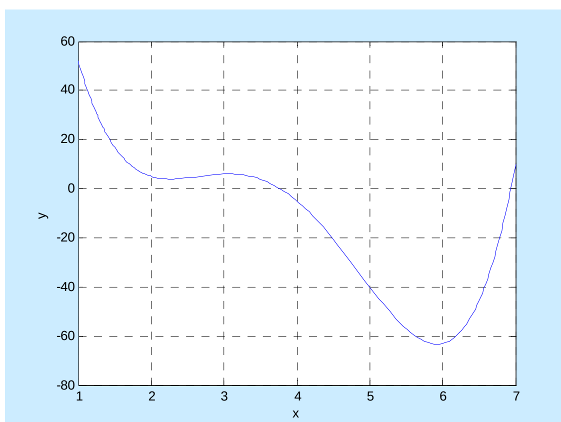
[d] 17. Which of the following is *NOT* a typical method for numerical differentiation?
a. The finite difference method
b. The Taylor series method
c. The method by curve fitting and analytic differentiation
d. The linear regression method

[c] 18. Which of the following strategies to reduce errors is incorrect in numerical differentiation/integration?
a. reducing h
b. using higher $O(h^n)$ method
c. using lower $O(h^n)$ method
d. using Richardson's extrapolation method

[a] 19. Which of the following is *NOT* a typical method for numerical solutions for ODEs with boundary values (BVs)?
a. Euler's implicit method
b. Reducing an ODE with BVs to a system of ODEs with IVs
c. The shooting methods
d. The finite difference method

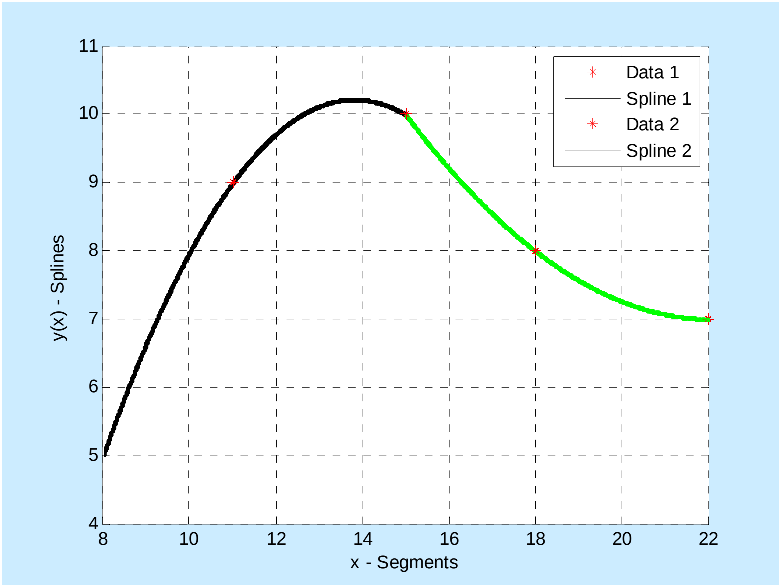
[b] 20. Given a polynomial fitted curve as shown in the figure, what are the approximate interpolation at $x = 2$ and the minimum of the function in $x \in [1, 4]$?

a. [5 -63]
b. [5 -7]
c. [2 -63]
d. [2 -7]



2. (20) Curve Fitting, Interpolation, and Integration

The spline fittings of a set of experimental results are shown in the figure by two 2nd order polynomials $f_1(x) = -0.15x^2 + 4.3x - 19.3$ and $f_2(x) = 0.1\,x^2 - 2.6x + 36.1$.



a. (5) Determine the interpolation y(x | x = 9) according to the 1st order Largrange interpolation method.
[Answer]

$$f(x\,|\,x=9) = \frac{(x-x_2)}{(x_1-x_2)}y_1 + \frac{(x-x_1)}{(x_2-x_1)}y_2$$
$$= \frac{(9-10)}{(8-10)}5 + \frac{9-8}{10-8}8$$
$$= 2.5+4$$
$$= 6.5$$

b. (6) Verify your result in Question (a) in term of the relative error with the actual analytic result of the given function.
[Answer]

$$f_1(x\,|\,x=9) = -0.15x^2 + 4.3x - 19.3$$
$$= -0.15*81 + 4.3*9 - 19.3$$
$$= -12.2 + 38.7 - 19.3$$
$$= 7.2$$
$$E_r = \left|\frac{7.2-6.5}{7.2}\right| = 9.72\%$$

c. (9) Use the *Gauss quadrature* integration method to calculate the numerical integration of the spline 1, i.e., $f_1(x)$ in [8, 15].
[Answer]

$$I = \int_a^b f(x)dx \simeq \int_{-1}^{1} f(\frac{g \bullet (b-a)+a+b}{2})(\frac{(b-a)}{2})dg$$

$$\simeq (\frac{(b-a)}{2})\sum_{i=1}^{2} \widehat{f}(\frac{g_i \bullet (b-a)+a+b}{2}), g_1 = -0.58, g_2 = 0.58$$

$$I = \int_8^{15} f(x)dx,\ f(x) = -0.15x^2 + \ 4.3x - 19.3$$

$$= 3.5\int_{-1}^{1} f(3.5g + 11.5)dg$$

$$= 3.5\sum_{i=1}^{2} \widehat{f}(3.5g_i + 11.5)$$

$$= 3.5\{[-0.15(-3.5*0.58+11.5)^2 + \ 4.3(-3.5*0.58+11.5)-19.3]+$$
$$[-0.15(3.5*0.58+11.5)^2 + \ 4.3(3.5*0.58+11.5)-19.3]\}$$

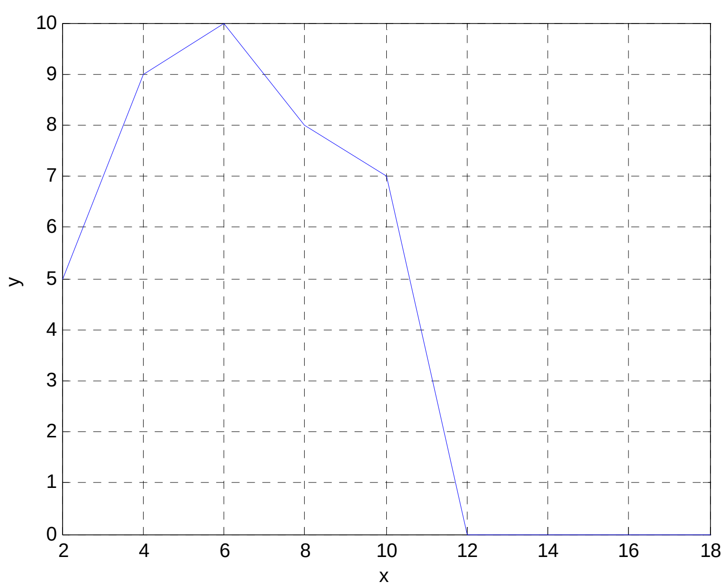
$$= 3.5[-13.5+40.7-19.3-27.5+58.1-19.3]$$

$$= 3.5(19.2)$$

$$= 67.2$$

3. (15) Numerical Differentiation and Integration

A set of data has been collected from an experiment as follows: *x* = [2, 4, 6, 8, 10, 12, 14, …, 10000] and *y* = [5, 9, 10, 8, 7, 0, 0, … , 0]. The data are plotted in the following figure.



a) (5) Determine where the curve's derivatives reached the maximum, minimum, and the turning point using the *2-point backward difference* method.

[Answer]

$$\frac{dy}{dx} = \frac{f(x_i) - f(x_{i-1})}{h}$$
$$= 0.5[f(4) - f(2),\ f(6) - f(4),\ f(8) - f(6),\ f(10) - f(8),\ f(12) - f(10),\ f(14) - f(12), ...]$$
$$= 0.5[4,\ 1,\ -2,\ -1,\ -7,\ 0,\ ...,\ 0]$$
$$= [2,\ 0.5,\ -1,\ -0.5,\ -3.5,\ 0,\ ...,\ 0]$$

That is:

$$D_{\max} = \frac{dy(4)}{dx} = 2.0$$
$$D_{\min} = \frac{dy(12)}{dx} = -3.5$$
$$D_o = \frac{dy(6)}{dx} \approx 0$$

b. (4) Calculate the numerical integration of this curve using the *trapezoid method*.

[Answer]

$$I = \frac{h}{2}[f(a) + f(b)] + h\sum_{i=2}^{\mathbf{n-1}} f(x_i)$$
$$= 5 + 0 + 2(9 + 10 + 8 + 7)$$
$$= 73$$

c. (6) Calculate the numerical integration of this curve using the *Simpson's 3/8* method and *Simpson's 1/3* method, respectively, for the first spline (4 points) and second spline (3 points) of the curve.
[Answer]

$$I_1 = \frac{3h}{8}\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right)$$
$$= \frac{3}{4}\left(5 + 3*9 + 3*10 + 8\right)$$
$$= \frac{3}{4}(70)$$
$$= 52.5$$
$$I_2 = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$
$$= \frac{2}{3}(8 + 4*7 + 0)$$
$$= 24$$
$$I = I_1 + I_2 = 52.5 + 24$$
$$= 76.5$$

4. (20) ODE with Initial Values and Boundary Values

Given an ODE $\frac{d^2y}{dx^2} = e^x,\ 0 \le x \le 2,\ h = 1$ with boundary values y(0) = 1 and y(2) = e^2 =7.3891, solve the following boundary value and initial value problems.

a. (5) Express the ODE as a system of first order ODEs and determine their IVs:
[Answer]

$$\frac{d^2y}{dx^2} = e^x \Rightarrow \begin{cases} \frac{dy}{dx} = w,\ \text{y}(0) = 1 \\ \frac{dw}{dx} = e^x,\ w(0) = 1 \end{cases}$$

[Observe e^x as a special function as in Q10.]

b. (5) Use *Euler's explicit* method to solve *w(x)* and *y(x)* for *x* = [1, 2].
[Answer]

$y_{i+1} = y_i + f(x_i, y_i)h = y_i + e^{x_i}$

(1) $x_0 = 0,\ w_0 = 1,\qquad y(0) = w(0) = 1$

(2) $x_1 = 1,\ w_1 = w_0 + e^{x_0} = 1 + e^0 = 2, y(1) = w(1) = 2$

(3) $x_2 = 2,\ w_2 = w_1 + e^{x_1} = 2 + e,\qquad y(2) = w(2) = 2 + e$

c. (5) Find the numerical solutions of *w(x)* for *x* = [1, 2] by the *modified Euler's* method, where the result obtained in question (b) may be reused.
[Answer]

$$\begin{cases} y_{i+1}^{Eu} = y_i + f(x_i, y_i)h = y_i + e^{x_i} \\ y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{Eu})}{2}h = y_i + 0.5(e^{x_i} + e^{x_{i+1}}) \end{cases}$$

(1) $x_0 = 0,\ w_0 = 1$

(2) $x_1 = 1,\ w_1 = w_0 + 0.5(e^{x_0} + e^{x_1}) = 1 + 0.5(1 + e) = 1.5 + 0.5e$

(3) $x_2 = 2,\ w_2 = w_1 + 0.5(e^{x_1} + e^{x_2}) = (1.5 + 0.5e) + 0.5(e + e^2) = 1.5 + e + 0.5e^2$

d. (5) Derive the function *y(x)* and determine its exact value at *x* = [1, 2].

[Answer]

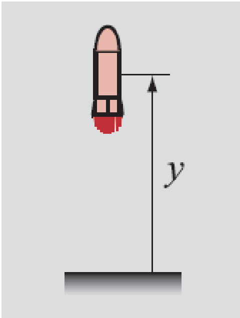
(i) $y(x) = \frac{dw}{dx} = e^x$

(ii) For $x = [1, 2],\ y(x) = [e, e^2]$

5. (15) ODE and Differentiations

An experimental rocket having an initial weight w = 3000 lb (including 2400 lb of fuel), and initially at rest, is launched vertically upward. The rocket burns fuel at a constant rate of 80 lb/s, which provides a constant thrust, T, of 8000 lb. The instantaneous weight of the rocket is $w(t) = 3000 - 80t$ lb. The drag force, D, experienced by the rocket is given by $D = 0.005g(\frac{dy}{dt})^2$ lb, where y is distance in ft, and g = 32.2 ft/s^2. According to Newton's law, the equation of motion for the rocket is given by:

$$\frac{w}{g}\frac{d^2y}{dt^2} = T - w - D, y(0) = 0, \frac{dy(0)}{dt} = 0$$



a. (5) Reduce the second-order ODE to a system of two first-order ODEs with proper initial values.
[Answer]

$$\frac{dy}{dt} = v,\ y(0) = 0$$

$$\frac{dv}{dt} = \frac{g}{w}(T - w - 0.008gv^2),\ v(0) = 0$$

b. (5) Describe the function of velocity of the rocket based on the results of Part (a). Assume the observation of the rocket's positions in the first 4 seconds at t = [0, 1, 2, 3, 4] (s) are y = [0, 30, 100, 180, 190] (ft), numerically determine the instantaneous velocity $v(t)$ of the rocket using the *central finite difference* method.
[Answer]

$$v(t) = \frac{dy}{dt}$$

According to the central finite diference method: $v(i) = \frac{f(t_{i+1}) - f(t_{i-1})}{t_{i+1} - t_{i-1}}, i = 1, 2, 3$

Obtain: $v(0) = 0$

$$v(1) = \frac{100-0}{2-0} = 50$$

$$v(2) = \frac{180-30}{3-1} = 75$$

$$v(3) = \frac{190-100}{4-2} = 45$$

c. (5) Express the function of the acceleration of the rocket based on the results of Part (b). Given the same observation of the rocket's positions during t = [0, 1, 2, 3, 4] (s), numerically determine the instantaneous acceleration *a(t)* of the rocket using the *backward finite difference* method. The results obtained in Part (b) may be reused.
[Answer]

$$a(t) = \frac{dv}{dt} = \frac{g}{w}(T - w - 0.008gv^2), v(0) = 0$$

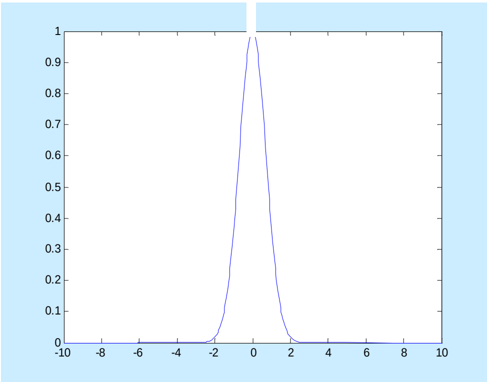
According to the backward finite difference method: $\mathrm{a}(i) = \frac{v(t_i) - v(t_{i-1})}{h}, i = 1, 2, 3$

Obtain: $a(0) = 0$

$$a(1) = 50 - 0 = 50$$
$$a(2) = 75 - 50 = 20$$
$$a(3) = 45 - 75 = -30$$

6. (10) Applied numerical integration in engineering

Given an engineering problem with the mathematical model $f(x) = e^{-x^2}$ as shown in the following figure, the integration of *f(x)* is $I = \int_{-\infty}^{\infty} e^{-x^2} dx = 1.7725$. Assuming that the value of the function $e^{-x^2}\big|_{x=0} = \infty$, solve the following problems:



a. (5) Describe your strategy and fomula(s) for analytically solving this integration problem?

$$I = \int_{-4}^{0-\varepsilon} e^{-x^2} dx + \int_{\varepsilon+0}^{4} e^{-x^2} dx$$

b. (5) Solve the problem by the Simpson's 1/3 method based on your strategy developed in Question (a). [Assume f(x | x=2) = 0.025]

$$
\begin{aligned}
I &= 2\frac{h}{3}\left(f\left(x_0\right)+4f\left(x_1\right)+f\left(x_2\right)\right)\\
&= \frac{4}{3}\left(f\left(0+\varepsilon\right)+4f\left(2\right)+f\left(4\right)\right)\\
&= \frac{4}{3}\left(1+4*0.025+0\right)\\
&= \frac{4}{3}*1.1000\\
&= 1.4667
\end{aligned}
$$

Aid Sheet

Derivative notation: $f^{(n)}(x) = \dfrac{d^n f(x)}{dx^n}$

Taylor expansion formula: $\widehat{f_N}(x) = \sum_{n=0}^{N} \dfrac{1}{n!} f^{(n)}(x_o)(x - x_o)^n$

The central finite difference method: $\dfrac{dy}{dx} = \dfrac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}}$

The midpoint integration method: $I = h\sum_{i=1}^{n} f(\dfrac{x_i + x_{i+1}}{2})$

The trapezoid integration method: $I = \dfrac{h}{2}[f(a) + f(b)] + h\sum_{i=2}^{\mathbf{n-1}} f(x_i)$

The 1st order Largrange interpolation: $f(x) = \dfrac{(x - x_2)}{(x_1 - x_2)} y_1 + \dfrac{(x - x_1)}{(x_2 - x_1)} y_2$

Simpson's $\frac{1}{3}$ integration method: $I = \dfrac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$

Simpson's $\frac{3}{8}$ integration method: $I = \dfrac{3h}{8}\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right)$

Euler's explicit method for IVP: $y_{i+1} = y_i + f(x_i, y_i)h$

The modified Euler's method for IVP: $y_{i+1} = y_i + \dfrac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{Eu})}{2} h$

Richardson's extrapolation method: $I(h \mid O(h^4)) = \dfrac{4I(h_2 \mid O(h_2{}^2) - I(h_1 \mid O(h_1{}^2)}{3}, \; h = h_2 = \dfrac{1}{2}h_1$

The predictor-corrector method for IVP:

$$\begin{cases} \text{The Predictor:} \quad y_{i+1}^{(1)} = y_i + f(x_i, y_i)h \\ \text{The Corrector:} \quad y_{i+1}^{(k)} = y_i + \dfrac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(k-1)})}{2} h \end{cases}$$

The Gauss quadrature integration:

$$I = \int_a^b f(x)dx \simeq \int_{-1}^{1} f(\frac{t \bullet (b-a) + a + b}{2})(\frac{(b-a)}{2})dt$$

$$\simeq (\frac{(b-a)}{2})\sum_{i=1}^{2} \hat{f}(\frac{t_i \bullet (b-a) + a + b}{2}) = \hat{f}\left(-\frac{1}{\sqrt{3}}\right) + \hat{f}\left(\frac{1}{\sqrt{3}}\right)$$