MAT1322, Final exam Practice sheet: partial solutions

Integration

Area, Volumes, Applications of Integrals

Compute the area of the following regions:

1. Bounded region delimited by $y = e^x, y = 1, x = 2$.
2. Bounded region delimited by $y = \sin(x), y = \cos(x), x = 0, y = 0, x = \frac{\pi}{2}$.
3. Bounded region delimited by $y = 3x - 6, y = 9/x, x = 4$.

Solution :

1. This region is from $x = 0$ to $x = 2$. Throughout this interval, the curve $y = e^x$ lies above $y = 1$. Thus, the area A is

$$A = \int_0^2 (e^x - 1)dx = \big[e^x - x\big]_0^2 = e^2 - 2 - (e^0 - 0) = e^2 - 3$$

2. This region consists of two parts: one from $x = 0$ to $x = \frac{\pi}{4}$ in which $y = \cos(x)$ lies above $y = \sin(x)$, and the other from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$ in which $y = \sin(x)$ lies above $y = \cos(x)$. Thus, the total area of these two regions is

$$\begin{aligned} A &= \int_0^{\pi/4} (\cos(x) - \sin(x))dx + \int_{\pi/4}^{\pi/2} (\sin(x) - \cos(x))dx \\ &= \big[\sin(x) + \cos(x)\big]_0^{\pi/4} + \big[-\cos(x) - \sin(x)\big]_{\pi/4}^{\pi/2} \\ &= (\sin(\pi/4) + \cos(\pi/4)) - (\sin(0) + \cos(0)) + (-\cos(\pi/2) - \sin(\pi/2)) - (-\cos(\pi/4) - \sin(\pi/4)) \\ &= \left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\right) - (0 + 1) + (0 - 1) - \left(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}\right) \\ &= 2\sqrt{2} - 2 \end{aligned}$$

3. This region spans the interval from $x = 3$ (the positive point of intersection of $y = 3x - 6$ and $y = 9/x$) to $x = 4$, and througout this interval, the line $y = 3x - 6$ lies above the curve $y = 9/x$. Thus, the area A is

$$A = \int_3^4 \left(3x - 6 - \tfrac{9}{x}\right) dx = \left[\tfrac{3x^2}{2} - 6x - 9\ln|x|\right]_3^4 = \left(\tfrac{3\cdot 16}{2} - 6(4) - 9\ln(4)\right) - \left(\tfrac{3\cdot 9}{2} - 6(3) - 9\ln(3)\right) \approx 1.91$$

Compute the volume of the following solids:

1. $y = e^x, y = 1, x = 2$ rotated around $y = -2$.

2. $y = \sin(x), y = \cos(x), x = 0, y = 0, x = \frac{\sqrt{2}}{2}$ about $y = -1$.
3. $y = 3x - 6, y = 9/x, y = 0, x = 4$ about $x = -2$.

Solution :

1. 133.0630975
2. 4.126940766
3. We use the method of cylindrical shells. The functions $y_1(x) = 3x - 6$ and $y_2(x) = 9/x$ intersect at -1 and 3, however we only need the positive intersect. The function y_1 crosses the x-axis at $x = 2$ (draw a sketch). The volume has to be compute in two parts: using y_1 from $x = 2$ to $x = 3$ and y_2 from $x = 3$ to $x = 4$. We rotate around the axis $x = -2$, hence the radius for the method of cylindrical shells becomes $r(x) = (x - (-2)) = x + 2$. Volume 1:

$$V_1 = 2\pi \int_2^3 (x+2)(3x-6)dx = 2\pi \int_2^3 3x^2 - 12dx = 14\pi = 43.98229716.$$

Volume 2:

$$V_2 = 2\pi \int_3^4 (x+2) \cdot \frac{9}{x} dx = 89.08474367.$$

So the total volume is $V = V_1 + V_2 = 133.0670408$.

Applications of integrals:

1. A cable that weighs $2kg/m$ is used to lift $800kg$ of coal up to a mine shaft which is $500m$ deep. Find the work done.

   Solution : The total work is the sum of the work from lifting the coal and the cable. We denote by x the depth from the top of the mine.

   The mass of a tiny segment of the cable of length Δx is $2\Delta x$ kg. Hence its weight is $2g\Delta x$ N, where g is the gravitational constant.

   Also the work required to lift the segment of cable of length Δx to the height x is

$$\Delta W = (2g\Delta x) \times x \text{ J}.$$

   Finally, the work required to lift the cable is

$$W_{\text{cable}} = \int_0^{500} 2gxdx = 2g \times \left[\frac{1}{2}x^2\right]_0^{500} = g \times 500^2 \simeq 2,45 \times 10^6 \text{ J}$$

   by using $g = 9.81$ m/s^2.

   The work to lift the coal is just

$$W_{\text{coal}} = 800 \times 500 \times 9.81 = 3924000 \text{ J}$$

   (since the force of the bag of coal does not change with the depth).

   The total work is hence

$$W = W_{\text{cable}} + W_{\text{coal}} \simeq 6374 \times 10^3 \text{ J}.$$

2. Suppose that 2 J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm. How much work is needed to stretch the spring from 35 cm to 40 cm? Hint: you may use Hooke's law, F=kx to compute k from the numbers given and then apply this to the question.

Solution : (corrected April 10) By Hooke's Law, we know that the force required to maintain a spring stretched x units beyond its natural length is proportional to x. That is, $f(x) = kx$. For this spring, its natural length is 0.3 m, which we will measure as $x = 0$. It requires 2 J to stretch this spring from $x = 0$ to $x = 0.12$ (meaning from 0.3 m to 0.42 m). Thus

$$2 = \int_0^{0.12} kx dx = \left[\frac{kx^2}{2}\right]_0^{0.12} = \frac{k(0.12)^2}{2} - \frac{k(0^2)}{2} = 0.0072k \qquad \Rightarrow \qquad k = \frac{2500}{9}$$

For the work required to stretch the spring from $x = 0.05$ (i.e. 0.35 m) to $x = 0.1$ (i.e. 0.40 m), we subdivide the interval $[0.05, 0.1]$ into n subintervals of length Δx.

The work to stretch the spring from x_i to $x_i + \Delta x$ is approximately

$$W_i \approx f(x_i)\Delta x = \frac{2500}{9} x_i \Delta x$$

The total work done to stretch the spring from 35 cm to 40 cm is thus

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2500}{9} x_i \Delta x = \int_{0.05}^{0.1} \frac{2500}{9} x dx = \left[\frac{2500x^2}{18}\right]_{0.05}^{0.1} = \frac{25}{24} \simeq 1.04 \text{ J}$$

3. A leaky 10 kg bucket is lifted from the ground to a height of 12 m at a constant speed with a rope that weighs 0.8 kg/m. Initially the bucket contains 36 kg of water, but the water leaks at a constant rate and finishes draining just as the bucket reaches the 12 m level. How much work is done?

Solution : (corrected April 11) We denote by x the height from the ground. Since we assume the loss of water is at a constant rate, the mass of water $M_w(x)$ in the bucket decreases linearly with x:

$$M_w(x) = 36 - kx \text{ l}$$

where k is a constant, and we know that $M_w(12) = 0$ l, we have

$$36 - 12k = 0 \text{ hence } 12k = 36 \text{ and } k = 3 \text{ kg/m.}$$

The mass of water contained in the bucket when it is at height x is hence

$$M_w(x) = 36 - 3x \text{ kg.}$$

The mass of rope $M_r(x)$ still hanging when the bucket is at a height of x m is

$$M_r(x) = (12 - x \text{ m})(0.8 \text{ kg/m}) = 9.6 - 0.8x \text{ kg}$$

Thus, when the bucket reaches height x m, the total force is

$$\begin{aligned} F(x) &= (\text{mass bucket} + \text{mass water} + \text{mass rope})g \\ &= (10 + 36 - 3x + 9.6 - 0.8x)g \\ &= (55.6 - 3.8x)g \text{ N} \end{aligned}$$

where $g \simeq 9.81$ m/s^2 is the acceleration of gravity.

Hence the work required to lift the bucket from height x to height $x + \Delta x$ is approximately

$$\Delta W(x) = (55.6 - 3.8x)g\Delta x \text{ J.}$$

In total the work required is

$$W = \int_0^{12} (55.6 - 3.8x)g dx = g\Big[55.6x - \frac{3.8}{2}x^2\Big]_0^{12} \simeq 3857 \text{ J.}$$

4. A tank with a capacity of 500 liters contains 500 liters of water with 100 kg of salt in solution. Water containing 1 kg of salt per liter is entering at the rate of 3 liters per minute, and the mixture is allowed to flow out of the tank at a rate of 3 liters per minute. Find the amount of salt in the tank after 5 minutes.

   Solution : let $C(t)$ denote the amount of salt in the container at time t. Then from the text we read that $C(0) = 100kg$. Further, the amount $C(t)$ changes as follows:

   (a) per minute we are losing $\frac{C(t)}{500} \cdot 3$ kg of salt, since at a time t there are exactly $C(t)/500$ kg of salt in one liter of the solution

   (b) At the same time, we are adding water with a salt concentration of $1kg/l$ and 3 liters per minute, so we are adding 3 kg of salt per minute.

   Hence we are adding 3 kg of salt per minute. This gives the following differential equation:

$$\frac{dC}{dt} = -\frac{3C}{500} + 3$$
$$\frac{dC}{dt} = \frac{-3C + 1500}{500}$$
$$\frac{dC}{-3C + 1500} = \frac{dt}{500}$$
$$\frac{dC}{3C - 1500} = -\frac{dt}{500}$$

   Which leads to

$$C(t) = K \cdot e^{-\frac{3t}{500}} + 500.$$

   Using the initial conditions $C(0) = 100$ we obtain $K = -400$ and so

$$C(t) = -400 \cdot e^{-\frac{3t}{500}} + 500.$$

   Hence to answer the question of the amount of salt after 5 minutes, we evaluate $C(t)$ at $t = 5$: $C(5) = -400 \cdot e^{-\frac{15}{500}} + 500 = 111.82$ kg.

5. A tank in the shape of an inverted cone has a height of 15 m and a base radius of 4 m and is filled with water to a depth of 12 meters. Determine the amount of work needed to pump all of the water to the top of the tank. Assume that the density of the water is $1000 kg/m^3$.

Solution : First, find the cross-section area function. Here the radius depends on the height x from the bottom of the tank. With similar triangles, we find that the radius at height x, denoted by $r(x)$ satisfies: $\frac{r(x)}{x} = \frac{4}{15}$, and so $r(x) = \frac{4x}{15}$.

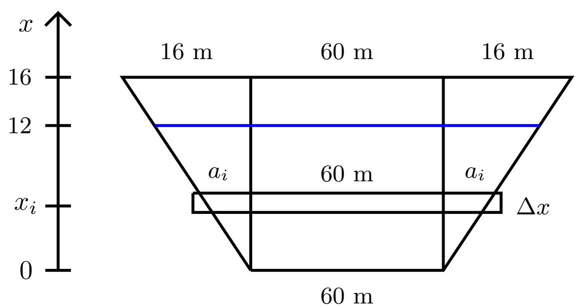
The amount of water at height x has to be lifted $15 - x$ meters. To pump the water to height 15 we set up the following integral:

$$W = \int_0^{12} \rho \cdot g \cdot \left(\frac{4x}{15}\right)^2 \pi \cdot (15 - x) dx.$$

Solve this with the usual methods from Calc I.

6. A dam has the shape of the trapezoid. The height is 16 meters and the width is 92 meters at the top, and 60 meters at the bottom. The water level is 4 meters below the top of the dam. Let x be the height of a horizontal stripe of the dam, measured from the bottom. Suppose ρ is the density of water, and g is the acceleration of gravity. Find the force acting on the dam (in Newtons).

Solution :



First, we approximate the force acting on a thin horizontal slice of the water at height x_i m from the bottom of the dam. To this end, let w_i denote the width of this strip and write $w_i = 60 + 2a_i$. By similar triangles, we see that

$$\frac{a_i}{16} = \frac{x_i}{16} \quad \Rightarrow \quad a_i = x_i \quad \Rightarrow \quad w_i = 60 + 2x_i$$

Thus, the area of this strip is $A_i \approx w_i \Delta x = (60 + 2x_i)\Delta x$

This strip is under a depth $d_i \approx 12 - x_i$ m of water, and so the force acting on this strip is approximately

$$F_i \approx \rho g A_i d_i = 9800(60 + 2x_i)(12 - x_i)\Delta x$$

Thus the total hydrostatic force acting on the face of the dam is

$$\begin{aligned} F &= \int_0^{12} 9800(60+2x)(12-x)dx \\ &= 9800 \int_0^{12} (720-36x-2x^2)dx \\ &= 9800 \left[720x - 18x^2 - \frac{2x^3}{3} \right]_0^{12} \\ &= 47980800 \ \ \text{N} \end{aligned}$$

Improper Integrals

Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

1. $\int_3^\infty \frac{1}{(x-2)^{3/2}} dx$.
2. $\int_1^5 \frac{1}{(x-2)^{2/3}} dx$.
3. $\int_1^\infty \frac{\cos^2(x^3)+4}{x^{2/3}} dx$
4. $\int_0^1 \frac{e^x}{x} dx$

Solution :

1. converges, value is 2.
2. converges, 1.326748710.
3. diverges, use comparison theorem and compare to $\int_1^\infty \frac{4}{x^{2/3}} dx$.
4. diverges, use comparison theorem and compare to $\int_0^1 \frac{1}{x} dx$.

Differential Equations

Use Euler's method to compute the following estimates:

1. Initial value problem $y' = y + 2x$ with $y(1) = 0$, take 4 approximation steps to estimate $y(2)$.
2. Step size $h = 0.1$ to estimate $y(0.3)$ of $y' = y + xy$ with $y(0) = 1$.

Solution :

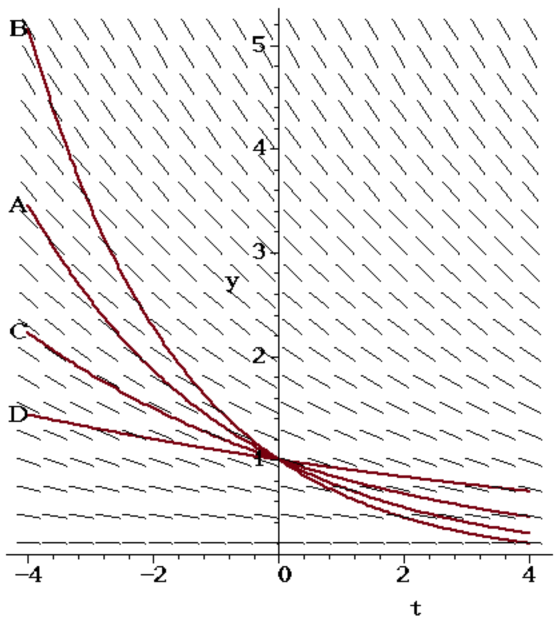
1. Step size has to be $h = 0.25$, approximate value for $y(2)$ is 3.765625.

2. Let $y' = f(x, y) = y + xy$.

$$\begin{array}{ll} x_0 = 0 & y_0 = 1 \\ x_1 = x_0 + h = 0.1 & y_1 = y_0 + f(x_0, y_0)h = 1 + (1 + (0)(1))(0.1) = 1.1 \\ x_2 = x_1 + h = 0.2 & y_2 = y_1 + f(x_1, y_1)h = 1.1 + (1.1 + 0.1 \cdot 1.1)(0.1) = 1.221 \\ x_3 = x_2 + h = 0.3 & y_3 = y_2 + f(x_2, y_2)h = 1.221 + (1.221 + 0.2(1.221))(0.1) = 1.36752 \end{array}$$

So the approximate value for $y(0.3)$ is 1.36752.

Direction fields:
We have sketched the slope field for the differential equation $\frac{dy}{dt} = F(t, y)$ in the graphic below. Which of the four curves (labeled A, B, C, D) that we have drawn over the slope field could be the solution to this differential equation with the initial condition $y(0) = 1$?



Solution : The answer is **C**. The slopes represent the direction of the tangent of a solution of the equation at each point: the curve of a solution has to be tangent to each slope it approaches.
Note: we can also observe from the slope fields that the solutions with image in the given interval (positive initial value) are decreasing.

Solve the initial value problem:

1. $\frac{dy}{dx} = xy^2, y(0) = 3.$
2. $xy^2y' = x + 1, y(1) = 1.$
3. $\frac{dy}{dx} = xe^{-y}, y(2) = 1.$

Solution :

1. $y(x) = \frac{1}{-\frac{x^2}{2}+\frac{1}{3}}$.
2. $y = \sqrt[3]{3x + 3\ln|x| - 2}$.
3. $y = \ln(\frac{x^2}{2} + (e-2))$.

Series

Tests for convergence:

(a) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$,

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$,

(c) $\sum_{k=1}^{\infty} \frac{1}{k^2+k^3}$,

(d) $\sum_{i=1}^{\infty} \frac{i}{\sqrt{i^5+1}}$,

(e) $\sum_{s=1}^{\infty} \frac{s^2-5s}{s^3+s+1}$,

(f) $\sum_{i=1}^{\infty} \frac{\ln(i)}{i}$,

(g) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n!}$,

(h) $\sum_{n=1}^{\infty} (-1)^{n+3} \frac{n^2}{n^3+4}$,

(i) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$,

(j) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

(k) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

(l) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$

(m) $\sum_{n=1}^{\infty} \ln(\frac{1}{n}+1)$

Solution :

(a) Convergent: use the integral test for $f(x) = x^2 e^{-x^3}$ (positive, decreasing function on $[1,\infty)$), one has $\int_1^{\infty} x^2 e^{-x^3} dx = \lim_{t\to\infty} \left(\frac{-e^{-t^3}}{3} + \frac{e^{-1}}{3} \right) = \frac{e^{-1}}{3}$ (convergent).

(b) Convergent: use the integral test for $f(x) = \frac{1}{x(\ln(x))^2}$ (positive, decreasing function on $[2,\infty)$), one has $\int_2^{\infty} \frac{1}{x(\ln(x))^2} dx = \lim_{t\to\infty} \left(\frac{-1}{\ln(t)} + \frac{1}{\ln(2)} \right) = \frac{1}{\ln(2)}$ (convergent).

(c) Convergent: compare to $\sum_{k=1}^{\infty} \frac{1}{k^3}$ (convergent),

(d) Convergent: compare to $\sum_{i=1}^{\infty} \frac{1}{i^{3/2}}$ (convergent),

(e) Divergent: compare to $\sum_{s=1}^{\infty} \frac{1}{s}$ (harmonic series, divergent),

(f) Divergent: this sum is larger than the harmonic series which is divergent (use comparison theorem).

(g) Convergent: this sum is smaller that $\sum_{n=1}^{\infty} \frac{e}{n!}$ which is convergent (by ratio test),

(h) Convergent by alternating series test: to prove that correctly, one needs to study the sequence $b_n = \frac{n^2}{n^3+4}$, show that it has limit 0 and that it is decreasing (note that the series is not absolutely convergent),

(i) Convergent by alternating series test (note that it is not absolutely convergent),

(j) (corrected April 11) divergent (use ratio test for example).

(k) Convergent (use root test for example).

(l) Convergent by alternating series test.

(m) Divergent: use the limit comparison theorem with $\frac{1}{n}$ (this strategy is inspired from MacLaurin series).

Series arithmetic

1. Find the partial sum s_3 of the series $\sum_{n=1}^{\infty} 1/n^4$. Estimate the error in using s_3 as an approximation to the sum of the series.
2. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ up to two decimal places. Remark: the calculation of the final sum may be longer than what will be asked in an exam.
3. Harder: Determine the sums of $\sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n)!}$, $\sum_{n=0}^{\infty} \frac{3^n}{2^n n!}$ and $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{2^{2n}}$. Hint: some may look like MacLaurin series of functions that you should remember. For the last series: try to split it into a positive and negative part to get two series that you recognise.
4. Use the MacLaurin Series of $\ln(x^2+1)$ to get a series with sum $\ln(1.25)$. How many terms of this series do we need to get an approximation with accuracy of 10^{-3}?

Solution :

1. $s_3 = \sum_{n=1}^{3} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} = \frac{1393}{1296} \simeq 1.0748457$. The error in using s_3 as an approximation of the series is $R_3 = \sum_{n=4}^{\infty} \frac{1}{n^4}$, this can be estimated using the integral test:

$$\int_4^{\infty} \frac{1}{x^4} dx \leq \sum_{n=4}^{\infty} \frac{(-1)^n}{n^2} \leq \int_3^{\infty} \frac{1}{x^4} dx$$
$$\frac{1}{3 \cdot 4^3} \leq R_3 \leq \frac{1}{3 \cdot 3^3}$$
$$0.005208 \leq R_3 \leq 0.012346$$

2. This is an alternating series, according to the theorem on the error of approximation of alternating series, the error of approximation of the series by the k-th partial derivative is smaller than the first term forgotten:

$$|R_k| = |\sum_{n=k+1}^{\infty} \frac{(-1)^n}{n^2}| \leq \frac{1}{(k+1)^2}.$$

To find the sum of the series up to two decimal places, we need to have an error smaller than 0.005.

$$\frac{1}{(k+1)^2} < \frac{1}{200}$$
$$k+1 > \sqrt{200}$$
$$k > 13.1$$

Taking $k = 14$, we obtain

$$s_{14} = \sum_{n=1}^{14} \frac{(-1)^n}{n^2} = -0.820$$

the value is correct up to two decimal places because the following term is negative: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \in [-0.825, -0.820]$.

3. The first one is $\cos(3)$, the second $e^{3/2}$, the third is a geometric series $\sum_{n=0}^{\infty} \frac{3}{(-4)^n}$, hence its sum is $\frac{3}{1-(-4)^{-1}} = \frac{12}{5}$.

4. We compute the MacLaurin Series of $\ln(x+1)$. Note that $\ln(x+1) = \int \frac{1}{1-(-x)}$, and $\frac{1}{1-(-x)}$ is the sum of the geometric series $\sum_{n=0}^{\infty}(-x)^n$. Hence

$$\ln(x+1) = \int \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty} \int (-1)^n x^n = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}.$$

Now to get the MacLaurin Series of $\ln(x^2+1)$, we substitute x^2 into the expression above:

$$\ln(x^2+1) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x^2)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}$$

So now:

$$\ln(1.25) = \ln(0.5^2+1) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{0.5^{2n}}{n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{0.25^n}{n}.$$

This is an alternating series, so to estimate how many terms we need to get an accuracy of 10^{-3} we need to check, which term a_n is smaller than $10^{-3}/2$:

$$\frac{1}{4^n \cdot n} < \frac{1}{10^3 \cdot 2}$$

$$4^n \cdot n > 10^3 \cdot 2$$

which is definitely satisfied if $n > 2 \cdot 10^3$. A more accurate estimation can not be solved precisely with the methods we have.

Find radius and Interval of Convergence:

1. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$
2. $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$
3. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$
4. $\sum_{n=1}^{\infty} \frac{n\cdot(x+1)^n}{4^n}$

Solution :

1. The radius of convergence is $\frac{5}{2}$, the interval of convergence is $\big[-2,3\big)$.
2. The radius of convergence is 3, the interval of convergence is $\big[-3,3\big)$.
3. The radius of convergence is 1, the interval of convergence is $\big[1,3\big]$.
4. the radius of convergence is 4, the interval of convergence is $\big(-5,3\big)$.

Taylor and MacLaurin series

Give (quoting from textbook allowed!) the MacLaurin series and their radii of convergence for:

1. $\frac{1}{1-x}$
2. e^x
3. $\sin x$
4. $\cos x$
5. $\arctan x$
6. $\ln(1+x)$
7. $(1+x)^k$

Compute the MacLaurin series (by using the definition OR by subsituting into a known series) for the following functions:

1. e^{-2x^2}
2. $x \cdot \cos(x)$
3. $\cos(x^2)$
4. $x \cdot \cos(\frac{1}{2}x^2)$
5. $(1+2x)^{1/4}$

Solution :

1. $\sum_{n=0}^{\infty} \frac{(-2x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{2n}}{n!}$
2. $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$
3. $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$
4. $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{2^{2n} \cdot (2n)!}$
5. $= \sum_{n=0}^{\infty} \binom{1/4}{n} (2x)^n$

Use the MacLaurin series expansions to compute the following integrals:

1. $\int \sin(x^2) dx$
2. $\int \ln(1+x^2) dx$
3. $\int e^{3\sqrt{x}} dx$

Solution :

1. $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!(4n+3)}$
2. This requires two steps. The first one to find the MacLaurin series of $\ln(1+x^2)$ and then the second one to compute the integral. In total we get $\int \ln(1+x^2)dx = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{n \cdot (2n+1)}$.
3. $= \sum_{n=0}^{\infty} \frac{3^n \cdot x^{\frac{n}{2}+1}}{n! \cdot (\frac{n}{2}+1)} = \sum_{n=0}^{\infty} \frac{2 \cdot 3^n \cdot x^{\frac{n}{2}+1}}{n! \cdot (n+2)}$

What is the general formula of the MacLaurin series? What is the general formula for the Taylor Series at a point a?

Functions of several variables

Level Curves, Traces

Sketch the graph of the following functions (it is recommended to begin with determining the domain, the range, then sketching some well chosen level curves and traces).

1. $f(x, y) = xy + 2$.
2. $f(x, y) = \frac{1}{\sqrt{x^2+4y^2}}$.
3. $f(x, y) = \ln(y - 3x^2)$

Solution :

1. The domain is $\mathbf{R}^2$, and the range is $\mathbf{R}$. The curve of level k has equation $xy = k - 2$:
   - if $k \neq 2$, this is an hyperbola: $y = \frac{k-2}{x}$,
   - if $k = 2$ this is the reunion of the two lines $x = 0$ and $y = 0$.

   The trace of the graph in the yz plane has equation $z = 0 + 2$, this is the horizontal line at height 2. The trace of the graph in the plane $x = 1$ has equation $z = y + 2$.
2. The domain is $\mathbf{R}^2 - \{(0, 0)\}$, the range is $(0, \infty)$. The curve of level $k > 0$ has equation $x^2 + 4y^2 = \frac{1}{k^2}$, this is an ellipse passing through the points $(\pm\frac{1}{k}, 0)$ and $(0, \pm\frac{1}{2k})$. The trace of the graph in the yz plane has equation $z = \frac{1}{|2y|}$.
3. $f(x, y) = \ln(y - 3x^2)$ The domain is $\{(x, y) \in \mathbf{R}^2, y > 3x^2\}$ (above the parabola $y = 3x^2$). The range is $\mathbf{R}$. The curve of level k has equation $y = 3x^2 + e^k$ this is a parabola. The trace of the graph in the yz plane has equation $z = \ln(y)$. The trace of the graph in the plane $y = 1$ has equation $z = \ln(1 - 3x^2)$ (only defined for $|x| < \frac{1}{9}$, with maximum 0 at $x = 0$).

Partial Derivatives

Find all second partial derivatives of f at the given points.

1. $f(x, y) = xy\sin(y^2)$ at $(3, 2)$.
2. $f(x, y) = y^5 - 3x^2y$ at $(4, 1)$.
3. $f(x, y) = \frac{x}{(x^2+y)^2}$ at $(1, 1)$.
4. $f(x, y) = \sqrt{x}\ln(y^x)$ at $(1, 4)$.

Solution :

1. $f'_x(x, y) = y\sin(y^2)$, $f'_y(x, y) = x\sin(y^2) + 2xy^2\cos(y^2)$,

$$\begin{aligned} f''_{xx}(x, y) &= 0 & f''_{xx}(3, 2) &= 0 \\ f''_{xy}(x, y) &= \sin(y^2) + 2y^2\cos(y^2) & f''_{xy}(3, 2) &= \sin(4) + 8\cos(4) \\ f''_{yy}(x, y) &= x(6y\cos(y^2) - 4y^3\sin(y^2)) & f''_{yy}(3, 2) &= 36\cos(4) - 96\sin(4)). \end{aligned}$$

2. $f'_x(x,y) = -6xy$, $f'_y(x,y) = 5y^4 + -3x^2$,

$$f''_{xx}(x,y) = -6y \quad f''_{xx}(4,1) = -6$$
$$f''_{xy}(x,y) = -6x \quad f''_{xy}(4,1) = -24$$
$$f''_{yy}(x,y) = 20y^3 \quad f''_{yy}(4,1) = 20.$$

3. (corrected April 10) $f_x(x,y) = \frac{(1)(x^2+y)^2 - x(2)(x^2+y)^1(2x)}{(x^2+y)^4} = \frac{-3x^2+y}{(x^2+y)^3}$ $\quad f_y(x,y) = \frac{-2x}{(x^2+y)^3}$,

$$f_{xx}(x,y) = \frac{12x^3 - 12xy}{(x^2+y)^4} \quad f_{xx}(1,1) = 0$$
$$f_{xy}(x,y) = \frac{10x^2 - 2y}{(x^2+y)^4} \quad f_{xy}(1,1) = \frac{1}{2}$$
$$f_{yx}(x,y) = \frac{10x^2 - 2y}{(x^2+y)^4} \quad f_{yx}(1,1) = \frac{1}{2}$$
$$f_{yy}(x,y) = \frac{6x}{(x^2+y)^4} \quad f_{yy}(1,1) = \frac{3}{8}.$$

4. $f'_x(x,y) = \frac{3}{2}\sqrt{x}\ln(y)$, $f'_y(x,y) = \frac{x^{3/2}}{y}$,

$$f''_{xx}(x,y) = \frac{3}{4\sqrt{x}}\ln(y) \quad f''_{xx}(1,4) = \frac{3}{4}\ln(4)$$
$$f''_{xy}(x,y) = \frac{3\sqrt{x}}{2y} \quad f''_{xy}(1,4) = \frac{3}{8}$$
$$f''_{yy}(x,y) = -\frac{x^{3/2}}{y^2} \quad f''_{yy}(1,4) = -\frac{1}{16}.$$

1. Find $f_{zyx} = \frac{\partial f}{\partial x \partial y \partial z}$ of $f(x,y,z) = 3xyz + x^2y^3z^7$. Evaluate at $(1,1,3)$.
2. Find $f_{yyx} = \frac{\partial f}{\partial x \partial y \partial y}$ of $f(x,y,z) = x^{\frac{12}{y}} + \ln(xy) + \sqrt{yz}$. Evaluate at $(1,2,3)$.
3. Let $u = e^{r\theta}\sin\theta$, find $\dfrac{\partial^3 u}{\partial r^2 \partial \theta}$.

Solution :

1.
$$f_z = \frac{\partial f}{\partial z} = 3xy + 7x^2y^3z^6.$$
$$f_{yz} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}[3xy + 7x^2y^3z^6] = 3x + 21x^2y^2z^6.$$
$$f_{xyz} = \frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}[3x + 21x^2y^2z^6] = 3 + 42xy^2z^6.$$

Thus
$$f_{xyz}(1,1,3) = 3 + 42(1)(1)^2(3)^6 = 30621.$$

2.

$$f_y = \frac{\partial f}{\partial y} = \frac{-12\ln(x)}{y^2} x^{\frac{12}{y}} + \frac{1}{y} + \frac{z}{2\sqrt{yz}}.$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}[\frac{-12\ln(x)}{y^2} x^{\frac{12}{y}} + \frac{1}{y} + \frac{z}{2\sqrt{yz}}] = \frac{24\ln(x)}{y^3} x^{\frac{12}{y}} + \frac{144(\ln(x))^2}{y^4} x^{\frac{12}{y}} - \frac{1}{y^2} - \frac{z^2}{4(yz)^{3/2}}.$$

$$\begin{aligned} f_{yyx} = \frac{\partial^3 f}{\partial x \partial y \partial y} &= \frac{\partial}{\partial x}[\frac{24\ln(x)}{y^3} x^{\frac{12}{y}} + \frac{144(\ln(x))^2}{y^4} x^{\frac{12}{y}} - \frac{1}{y^2} - \frac{z^2}{4(yz)^{3/2}}] \\ &= \frac{24}{y^3} x^{\frac{12}{y}-1} + \frac{288\ln(x)}{y^4} x^{\frac{12}{y}-1} + \frac{288\ln(x)}{y^4} x^{\frac{12}{y}-1} + \frac{1728(\ln(x))^2}{y^5} x^{\frac{12}{y}-1}. \end{aligned}$$

Thus

$$f_{yyx}(1,2,3) = \frac{24}{2^3} 1^{\frac{12}{2}-1} + 0 = 3.$$

3.

$$\frac{\partial u}{\partial \theta} = re^{r\theta} \sin\theta + e^{r\theta} \cos\theta.$$

$$\begin{aligned} \frac{\partial^2 u}{\partial r \partial \theta} &= \frac{\partial}{\partial r}[re^{r\theta} \sin\theta + e^{r\theta} \cos\theta] \\ &= \left(1 \cdot e^{r\theta} + re^{r\theta} \cdot \theta\right) \sin\theta + \theta e^{r\theta} \cos\theta \\ &= e^{r\theta} \left((1 + r\theta)\sin\theta + \theta\cos\theta\right) \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial^3 u}{\partial r^2 \partial \theta} &= \frac{\partial}{\partial r}[e^{r\theta} \left((1 + r\theta)\sin\theta + \theta\cos\theta\right)] \\ &= e^{r\theta} \cdot \theta \cdot \left((1 + r\theta)\sin\theta + \theta\cos\theta\right) + e^{r\theta}(\theta \sin\theta) \\ &= \theta e^{r\theta} \left((2 + r\theta)\sin\theta + \theta\cos\theta\right). \end{aligned}$$

Linear approximation:

1. Determine the linear approximation of the function $f(x, y) = xy\sin(y^2)$ at the point $(3, 2)$. Use this linear approximation to give an estimate for the value of $f(3.1, 2.02)$.
2. Determine the linear approximation of the function $f(x, y, z) = 3xyz + x^2y^3z^7$ at the point $(1, 1, 3)$. Use this linear approximation to give an estimate for the value of $f(0.9, 0.8, 3.01)$.

Solution :

1. The equation of the tangent plane to the surface $z = f(x, y)$ at the point $(3, 2)$ is

$$L(x, y) = f(3, 2) + \frac{\partial f}{\partial x}(3, 2)(x - 3) + \frac{\partial f}{\partial y}(3, 2)(y - 2)$$

where $\frac{\partial f}{\partial x} = y\sin(y^2)$ and $\frac{\partial f}{\partial y} = x\sin(y^2) + 2xy^2\cos(y^2)$. Thus,

$$\begin{aligned} L(x,y) &= 6\sin(4) + 2\sin(4)(x-3) + (3\sin(4) + 24\cos(4))(y-2) \\ &\simeq -4.54 - 1.51(x-3) - 17.96(y-2). \end{aligned}$$

Thus,

$$f(3.1, 2.02) \approx L(3.1, 2.02) = -4.54 - 1.51(3.1-3) - 17.96(2.02-2) = -5.05.$$

2.

$$L(x,y,z) = f(1,1,3) + \frac{\partial f}{\partial x}(1,1,3)(x-1) + \frac{\partial f}{\partial y}(1,1,3)(y-1) + \frac{\partial f}{\partial z}(1,1,3)(z-3)$$

where $\frac{\partial f}{\partial x} = 3yz + 2xy^3z^7$, $\frac{\partial f}{\partial y} = 3xz + 3x^2y^2z^7$, and $\frac{\partial f}{\partial z} = 3xy + 7x^2y^3z^6$. Thus,

$$\begin{aligned} L(x,y,z) &= 2196 + 4383(x-1) + 6570(y-1) + 5106(z-3) \\ &= 4383x + 6570y + 5106z - 24075. \end{aligned}$$

Thus,

$$f(0.9, 0.8, 3.01) \approx L(0.9, 0.8, 3.01) = 4383(0.9) + 6570(0.8) + 5106(3.01) - 24075 = 494.76.$$

Chain rule

1. $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z = x^4 + x^2y$, $x = s + 2t$, $y = st$
2. $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ for $f(x,y) = \frac{x+y}{x-y}$, $x = 3st^2$, $y = s^2 + t^2$
3. For a function $z = (g(x), h(x))$ find $z'(7)$ with the given values:

$$g(7) = 3, \quad h(7) = 2, \quad g'(7) = -4, \quad h'(7) = 1, \quad \frac{\partial z}{\partial g}(3,2) = -2, \quad \frac{\partial z}{\partial h}(3,2) = 9.$$

4. For a function $z = (t(u,v), s(u,v))$ find $\frac{\partial z}{\partial u}(2,3)$ with the given values:

$$t(2,3) = 4, \quad s(2,3) = 6, \quad \frac{\partial t}{\partial u}(2,3) = -4, \quad \frac{\partial s}{\partial u}(2,3) = 1, \quad \frac{\partial z}{\partial t}(4,6) = -2, \quad \frac{\partial z}{\partial s}(4,6) = 8.$$

Solution :

1.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (4x^3 + 2xy)(1) + (x^2)(t).$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (4x^3 + 2xy)(2) + (x^2)(s).$$

2.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{-2y}{(x-y)^2}(2t^2) + \frac{2x}{(x-y)^2}(2s).$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{-2y}{(x-y)^2}(6st) + \frac{2x}{(x-y)^2}(2t).$$

3.

$$z'(x) = \frac{\partial z}{\partial g}(g(x), h(x))g'(x) + \frac{\partial z}{\partial h}(g(x), h(x))h'(x)$$

Thus,

$$\begin{aligned} z'(7) &= \frac{\partial z}{\partial g}(g(7), h(7))g'(7) + \frac{\partial z}{\partial h}(g(7), h(7))h'(7) \\ &= \frac{\partial z}{\partial g}(3,2)(-4) + \frac{\partial z}{\partial h}(3,2)(1) = (-2)(-4) + 9(1) = 17. \end{aligned}$$

4.

$$\frac{\partial z}{\partial u}(u,v) = \frac{\partial z}{\partial t}(t(u,v), s(u,v))\frac{\partial t}{\partial u}(u,v) + \frac{\partial z}{\partial s}(t(u,v), s(u,v))\frac{\partial s}{\partial u}(u,v)$$

Thus,

$$\begin{aligned} \frac{\partial z}{\partial u}(2,3) &= \frac{\partial z}{\partial t}(t(2,3), s(2,3))\frac{\partial t}{\partial u}(2,3) + \frac{\partial z}{\partial s}(t(2,3), s(2,3))\frac{\partial s}{\partial u}(2,3) \\ &= \frac{\partial z}{\partial t}(4,6)(-4) + \frac{\partial z}{\partial s}(4,6)1 = (-2)(-4) + 8(1) = 16. \end{aligned}$$

Implicit differentiation and gradient vectors

Consider the function $z = f(x, y)$ defined implicitly by the equation $x^2z + xy - yz^3 = -1$.

1. Find the gradient vector of the function $z = f(x, y)$ at the point $(2, 1, -1)$.
2. Find the equation of the tangent plane of the graph of the equation at the point $(2, 1, -1)$.
3. Find the directional derivative of this function at the point $(2, 1, -1)$, in the direction of the vector $\mathbf{v} = (2, -3)$.
4. Find the maximum value of the directional derivative at $(2, 1, -1)$ among all possible directions.

Solution :

1.

$$\nabla f(2,1) = \left(\frac{\partial z}{\partial x}(2,1,-1), \frac{\partial z}{\partial y}(2,1,-1)\right).$$

First we calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$2xz + x^2\frac{\partial z}{\partial x} + y - 3yz^2\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}[x^2 - 3yz^2] = -y - 2xz \quad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-y-2xz}{x^2-3yz^2}.$$

Thus,

$$\frac{\partial z}{\partial x}(2,1,-1) = \frac{-1-2(2)(-1)}{2^2-3(1)(-1)^2} = 3.$$

Similarly,

$$x^2\frac{\partial z}{\partial y} + x - z^3 - 3yz^2\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y}[x^2 - 3yz^2] = z^3 - x \quad \Rightarrow \quad \frac{\partial z}{\partial y} = \frac{z^3-x}{x^2-3yz^2}.$$

Thus,

$$\frac{\partial z}{\partial y}(2,1,-1) = \frac{(-1)^3-2}{2^2-3(1)(-1)^2} = -3.$$

Thus,

$$\nabla f(2,1) = (3,-3)\,.$$

2. The equation of the tangent plane to the surface $z = f(x,y)$ at the point $(2,1,-1)$ is

$$\begin{aligned} z &= -1 + \frac{\partial z}{\partial x}(2,1,-1)(x-2) + \frac{\partial z}{\partial y}(2,1,-1)(y-1) \\ &= -1 + 3(x-2) - 3(y-1) = 3x - 3y - 4. \end{aligned}$$

3. Note that v is not a unit vector, but since $|v| = \sqrt{13}$, the unit vector in the direction of v is

$$u = \frac{v}{|v|} = (\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}})$$

Therefore,

$$D_u f(2,1) = \nabla f(2,1).u = (3,-3)\cdot(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}) = \frac{6}{\sqrt{13}} - \frac{9}{\sqrt{13}} = -\frac{3}{\sqrt{13}}.$$

4. The maximum rate of change is $|\nabla f(2,1)| = |<3,-3>| = \sqrt{3^2+(-3)^2} = \sqrt{18} = 3\sqrt{2}$.

Directional Derivatives

Beware: the directional vector may need to be of a certain length.

1. $f(x,y) = e^x \sin(y)$, $u = \langle 1,5\rangle$ at $(0, \frac{\pi}{4})$
2. $f(x,y) = \frac{x}{x^2+y^2}$, in the direction which forms the angle $\theta = \pi/3$ with the x-axis, evaluate at $(2,4)$
3. $f(x,y) = x^2y - xy$ from $u = 3\cdot\mathbf{i} + 6\cdot\mathbf{j}$. Recall that $\mathbf{i} = \langle 1,0\rangle$ and $\mathbf{j} = \langle 0,1\rangle$.

Solution :

1. We have $f_x = e^x \sin(y)$ and $f_y = e^x \cos(y)$. Thus, we first calculate the gradient vector at $(0, \frac{\pi}{4})$

$$\nabla f(x, y) = e^x \sin(y)i + e^x \cos(y)j \Rightarrow \nabla f(0, \frac{\pi}{4}) = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j.$$

Note that u is not a unit vector, but since $|u| = \sqrt{26}$, the unit vector in the direction of u is

$$v = \frac{u}{|u|} = \frac{1}{\sqrt{26}}i + \frac{5}{\sqrt{26}}j$$

Therefore,

$$D_v f(0, \frac{\pi}{4}) = \nabla f(0, \frac{\pi}{4}).v = (\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j) \cdot (\frac{1}{\sqrt{26}}i + \frac{5}{\sqrt{26}}j) = \frac{\sqrt{2}}{2\sqrt{26}} + \frac{5\sqrt{2}}{2\sqrt{26}} = \frac{6\sqrt{2}}{2\sqrt{26}} = \frac{3}{\sqrt{13}}.$$

2. Since the direction we are interested in makes an angle $\theta = \pi/3$ with the x-axis, the unit vector we are looking for is

$$\mathbf{u} = \langle \cos\theta, \sin\theta \rangle = \langle \cos(\pi/3), \sin(\pi/3) \rangle = \left\langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \right\rangle$$

We have $f_x = \frac{(1)(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$ and $f_y = \frac{(0)(x^2+y^2)-x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$.
The gradient vector evaluated at the point $(2, 4)$ is thus

$$\nabla f(2, 4) = \langle f_x(2, 4), f_y(2, 4) \rangle = \left\langle \tfrac{12}{400}, \tfrac{16}{400} \right\rangle = \left\langle \tfrac{3}{100}, \tfrac{1}{25} \right\rangle$$

Therefore, the directional derivative we want is given by

$$D_{\mathbf{u}} f(2, 4) = \nabla f(2, 4) \cdot \mathbf{u} = \left\langle \tfrac{3}{100}, \tfrac{1}{25} \right\rangle \cdot \left\langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \right\rangle = \tfrac{3}{200} + \tfrac{\sqrt{3}}{50}$$

3. We have $f_x = 2xy - y$ and $f_y = x^2 - x$. Thus, the gradient vector is

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 2xy - y, x^2 - x \rangle$$

Note that $\mathbf{u}$ is not a unit vector, but $|\mathbf{u}| = \sqrt{3^2 + 6^2} = \sqrt{45}$. Thus, the unit vector in the direction of $\mathbf{u} = 3\mathbf{i} + 6\mathbf{j} = \langle 3, 6 \rangle$ is

$$\hat{u} = \frac{|\mathbf{v}|}{||\mathbf{v}||} = \frac{\langle 3, 6 \rangle}{\sqrt{45}} = \left\langle \frac{3}{\sqrt{45}}, \frac{6}{\sqrt{45}} \right\rangle$$

Therefore, the directional derivative is

$$D_{\hat{\mathbf{u}}} f(x, y) = \nabla f \cdot \hat{\mathbf{u}} = \langle 2xy - y, x^2 - x \rangle \cdot \left\langle \tfrac{3}{\sqrt{45}}, \tfrac{6}{\sqrt{45}} \right\rangle = \tfrac{3}{\sqrt{45}}(2xy - y) + \tfrac{6}{\sqrt{45}}(x^2 - x)$$

Good luck with studying!