ELG3175
Introduction to Communication Systems

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Solutions

Assignment #5

#1 Consider "demodulator" operation in details.

$$x_1(t) = x^3(t) = A_c^3 m^3(t) \cos^3 \omega_c t .$$

Using $\cos^3 \alpha = \cos\alpha \cdot \frac{1}{2}(1+\cos 2\alpha) = \frac{3}{4}\cos\alpha + \frac{1}{4}\cos 3\alpha$

one obtains:

$$x_1(t) = \frac{3}{4} A_c^3 m^3(t) \cos \omega_c t + \frac{1}{4} A_c^3 m^3(t) \cos 3\omega_c t$$

BPF at $f_0 = 3f_c$ rejects all terms except for $\cos 3\omega_c t$ term:

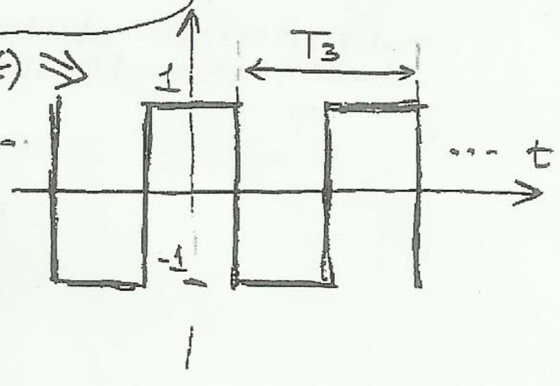
$$x_2(t) = \frac{1}{4} A_c^3 m^3(t) \cos 3\omega_c t$$

Ideal limiter produces the following:

$$x_3(t) = \operatorname{sgn}[m(t)] \underbrace{\left[2 \sum_{n=-\infty}^{+\infty} \Pi\left(\frac{2}{T_3}t - 2n\right) - 1 \right]}_{s(t)}, \quad T_3 = \frac{1}{3f_c} = \frac{2\pi}{3\omega_c}$$

Note that we must account for the sign of $m(t)$ (i.e., $\operatorname{sgn}[m(t)] = \pm 1$, depending on $m(t) \gtrless 0$).

$s(t) \Rightarrow$



the pulse train $s(t)$ can be expanded in Fourier series:

$$s(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos(2k+1)\omega_3 t = \frac{4}{\pi}\cos\omega_3 t + \text{high-frequency terms}$$

$\omega_3 = 3\omega_c$. BPF at $f_0 = 3f_c$ rejects all terms except for $\cos 3\omega_c t$:

$$x_4(t) = \operatorname{sgn}[m(t)] \frac{4}{\pi} \cos 3\omega_c t$$

#1 Further,

$$x_5(t) = \text{sgn}[m(t)] \cdot \frac{4}{\pi} \cdot \cos \omega_c t$$

and

$$x_6(t) = \frac{4}{\pi} \text{sgn}[m(t)] \cos^2 \omega_c t \cdot A_c m(t) =$$

$$= \frac{4}{\pi} A_c |m(t)| \cdot \frac{1}{2}(1 + \cos 2\omega_c t).$$ Note that: $\text{sgn}[m(t)]\, m(t) = |m(t)|$

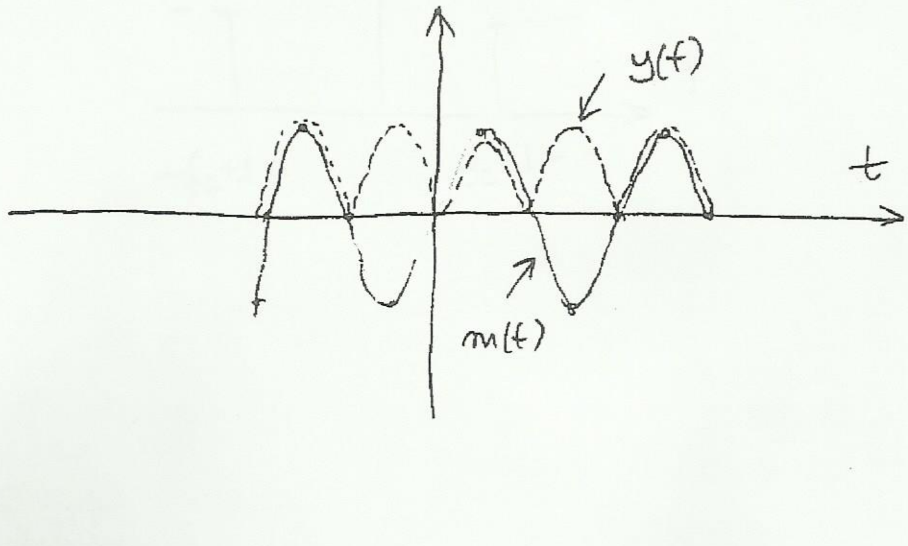
LPF rejects $\cos 2\omega_c t$ and

$$y(t) = \frac{2}{\pi} A_c \cdot |m(t)|$$

Hence, $y(t) \sim |m(t)|$ and we loose any information about polarity of $m(t)$!

It is not a demodulator $\rightarrow$ the loss of polarity cannot be recovered. For exampee, transmitting ± 1, we receive only $+1$!

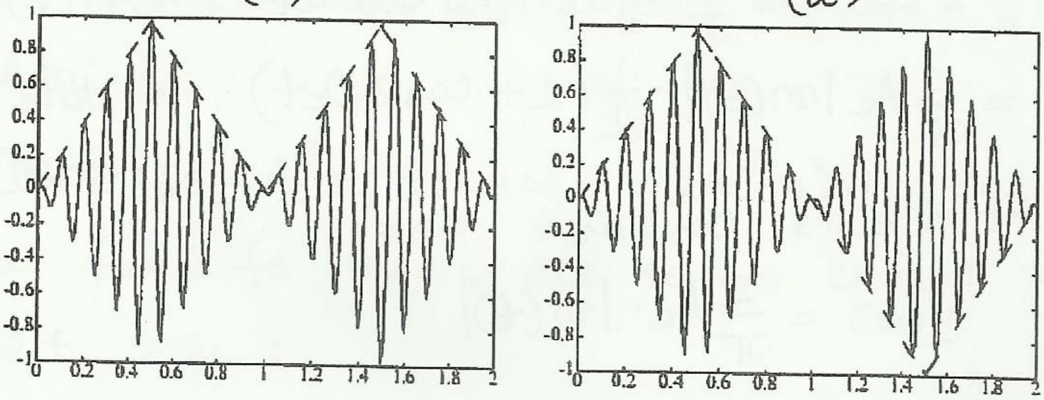
(2b) $m(t) = \sin \Omega t$; $y(t) = |\sin(\Omega t)|$



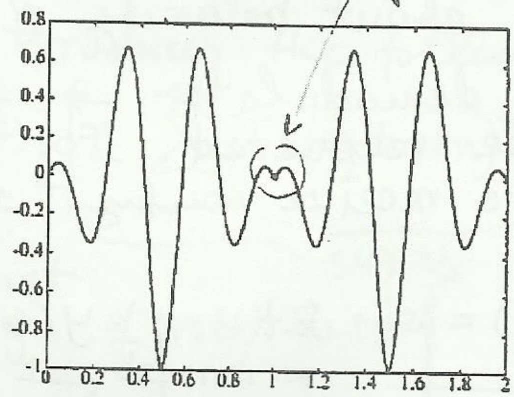
Problem [redacted] 2

The following figure shows the modulated signals for $A = 1$ and $f_0 = 10$. As it is observed both signals have the same envelope but there is a phase reversal at $t = 1$ for the second signal $Am_2(t)\cos(2\pi f_0 t)$ ([redacted] Left plot). This discontinuity is shown clearly in the next figure where we plotted $Am_2(t)\cos(2\pi f_0 t)$ with $f_0 = 3$.

(b) (a)



(a) → expanded!



Problem [redacted] 3

1) The Hilbert transform of $\cos(2\pi 1000t)$ is $\sin(2\pi 1000t)$, whereas the Hilbert transform of $\sin(2\pi 1000t)$ is $-\cos(2\pi 1000t)$. Thus

$$\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$$

2) The expression for the LSSB AM signal is

$$u_l(t) = A_c m(t)\cos(2\pi f_c t) + A_c \hat{m}(t)\sin(2\pi f_c t)$$

Substituting $A_c = 100$, $m(t) = \cos(2\pi 1000t) + 2\sin(2\pi 1000t)$ and $\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$ in the previous, we obtain

$$\begin{aligned}
u_l(t) &= 100\left[\cos(2\pi 1000t) + 2\sin(2\pi 1000t)\right]\cos(2\pi f_c t) \\
&+ 100\left[\sin(2\pi 1000t) - 2\cos(2\pi 1000t)\right]\sin(2\pi f_c t) \\
&= 100\left[\cos(2\pi 1000t)\cos(2\pi f_c t) + \sin(2\pi 1000t)\sin(2\pi f_c t)\right] \\
&+ 200\left[\cos(2\pi f_c t)\sin(2\pi 1000t) - \sin(2\pi f_c t)\cos(2\pi 1000t)\right] \\
&= 100\cos(2\pi(f_c - 1000)t) - 200\sin(2\pi(f_c - 1000)t) =
\end{aligned}$$

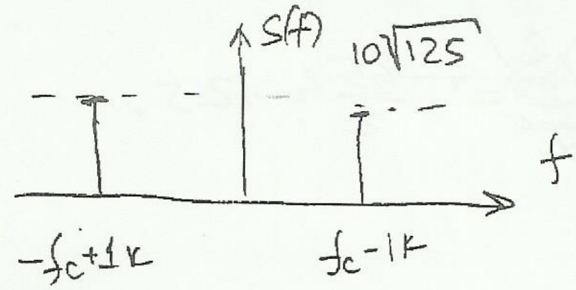
$$= \overbrace{\sqrt{100^2+200^2}}^{A}\left(\overbrace{\frac{100}{A}}^{\cos\varphi}\cos w_1 t - \overbrace{\frac{200}{A}}^{\sin\varphi}\sin w_1 t\right) = A\cos(w_1 t + \varphi)$$

3) Taking the Fourier transform of the previous expression we obtain

$$\begin{aligned}
U_l(f) &= 50\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right) \\
&+ 100j\left(\delta(f - f_c + 1000) - \delta(f + f_c - 1000)\right) \\
&= (50 + 100j)\delta(f - f_c + 1000) + (50 - 100j)\delta(f + f_c - 1000) = \frac{A}{2}\left(\delta(f-f_c)e^{j\varphi} + \delta(f+f_c)e^{-j\varphi}\right)
\end{aligned}$$

Hence, the magnitude spectrum is given by

$$\begin{aligned}
|U_l(f)| &= \sqrt{50^2 + 100^2}\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right) \\
&= 10\sqrt{125}\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right)
\end{aligned}$$



#4 a) $\hat{m}(t) = 5\cos(\omega_1 t - \pi/2) = 5\sin(\omega_1 t)$.

b) $s(t) = m(t)\cos\omega_c t + \hat{m}(t)\sin\omega_c t =$

$= 5\cos\omega_1 t\cos\omega_c t + 5\sin\omega_1 t\sin\omega_c t$

Using $\begin{cases} \cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right) \\ \sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right), \end{cases}$

$s(t) = 5\cos[(\omega_c - \omega_1)t]$

c) $V_{rms} = \frac{5}{\sqrt{2}}$;

d) $V_p = 5$;

e) $P_{av} = \lim_{T\to\infty} \frac{1}{T}\int_T s^2(t)dt = V_{rms}^2 = \frac{25}{2} = 12.5$.

f) $PEP = \frac{V_p^2}{2} = \frac{25}{2} = 12.5$.

Problem [redacted] 5

The mixed signal $y(t)$ is given by

$$\begin{aligned} y(t) &= u(t) \cdot x_L(t) = Am(t)\cos(2\pi f_c t)\cos(2\pi f_c t + \theta) \\ &= \frac{A}{2} m(t)\left[\cos(2\pi 2 f_c t + \theta) + \cos(\theta)\right] \end{aligned}$$

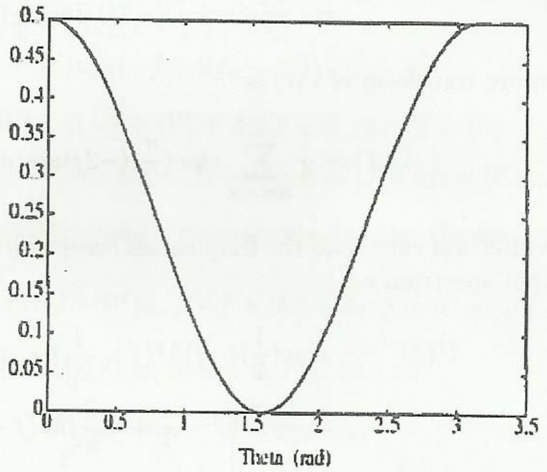
The lowpass filter will cut-off the frequencies above W, where W is the bandwidth of the message signal $m(t)$. Thus, the output of the lowpass filter is

$$z(t) = \frac{A}{2} m(t)\cos(\theta)$$

If the power of $m(t)$ is P_M, then the power of the output signal $z(t)$ is $P_{\text{out}} = P_M \frac{A^2}{4}\cos^2(\theta)$. The power of the modulated signal $u(t) = Am(t)\cos(2\pi f_c t)$ is $P_U = \frac{A^2}{2} P_M$. Hence,

$$\frac{P_{\text{out}}}{P_U} = \frac{1}{2}\cos^2(\theta)$$

A plot of $\frac{P_{\text{out}}}{P_U}$ for $0 \le \theta \le \pi$ is given in the next figure.



Problem 6

If we let

$$x(t) = -\Pi\left(\frac{t+\frac{T_p}{4}}{\frac{T_p}{2}}\right) + \Pi\left(\frac{t-\frac{T_p}{4}}{\frac{T_p}{2}}\right)$$

then using the results of Problem 2.23, we obtain

$$\begin{aligned} v(t) &= m(t)s(t) = m(t)\sum_{n=-\infty}^{\infty} x(t-nT_p) \\ &= m(t)\frac{1}{T_p}\sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_p}\right)e^{j2\pi\frac{n}{T_p}t} \end{aligned}$$

where

$$\begin{aligned} X\left(\frac{n}{T_p}\right) &= \mathcal{F}\left[-\Pi\left(\frac{t+\frac{T_p}{4}}{\frac{T_p}{2}}\right) + \Pi\left(\frac{t-\frac{T_p}{4}}{\frac{T_p}{2}}\right)\right]\Bigg|_{f=\frac{n}{T_p}} \\ &= \frac{T_p}{2}\operatorname{sinc}\left(f\frac{T_p}{2}\right)\left(e^{-j2\pi f\frac{T_p}{4}} - e^{j2\pi f\frac{T_p}{4}}\right)\Bigg|_{f=\frac{n}{T_p}} \\ &= \frac{T_p}{2}\operatorname{sinc}\left(\frac{n}{2}\right)(-2j)\sin\left(n\frac{\pi}{2}\right) \end{aligned}$$

Hence, the Fourier transform of $v(t)$ is

$$V(f) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\operatorname{sinc}\left(\frac{n}{2}\right)(-2j)\sin\left(n\frac{\pi}{2}\right)M\left(f-\frac{n}{T_p}\right)$$

The bandpass filter will cut-off all the frequencies except the ones centered at $\frac{1}{T_p}$, that is for $n = \pm 1$. Thus, the output spectrum is

$$\begin{aligned} U(f) &= \operatorname{sinc}\left(\frac{1}{2}\right)(-j)M\left(f-\frac{1}{T_p}\right) + \operatorname{sinc}\left(\frac{1}{2}\right)jM\left(f+\frac{1}{T_p}\right) \\ &= -\frac{2}{\pi}jM\left(f-\frac{1}{T_p}\right) + \frac{2}{\pi}jM\left(f+\frac{1}{T_p}\right) \\ &= \frac{4}{\pi}M(f) \star \left[\frac{1}{2j}\delta\left(f-\frac{1}{T_p}\right) - \frac{1}{2j}\delta\left(f+\frac{1}{T_p}\right)\right] \end{aligned}$$

Taking the inverse Fourier transform of the previous expression, we obtain

$$u(t) = \frac{4}{\pi}m(t)\sin\left(2\pi\frac{1}{T_p}t\right)$$

which has the form of a DSB-SC AM signal, with $c(t) = \frac{4}{\pi}\sin(2\pi\frac{1}{T_p}t)$ being the carrier signal.

Note: a different form of this solution was presented in the lecture #6.

Problem [redacted] 7

1) The spectrum of the modulated signal $Am(t)\cos(2\pi f_c t)$ is

$$V(f) = \frac{A}{2}[M(f-f_c) + M(f+f_c)]$$

The spectrum of the signal at the output of the highpass filter is

$$U(f) = \frac{A}{2}[M(f+f_c)u_{-1}(-f-f_c) + M(f-f_c)u_{-1}(f-f_c)]$$

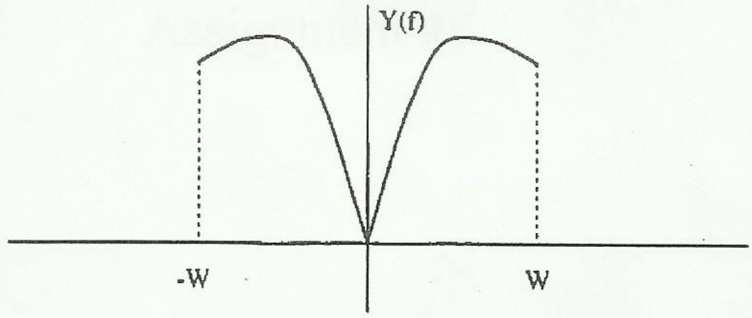
Multiplying the output of the HPF with $A\cos(2\pi(f_c+W)t)$ results in the signal $z(t)$ with spectrum

$$\begin{aligned}
Z(f) &= \frac{A}{2}[M(f+f_c)u_{-1}(-f-f_c) + M(f-f_c)u_{-1}(f-f_c)] \\
&\quad \star \frac{A}{2}[\delta(f-(f_c+W)) + \delta(f+f_c+W)] \\
&= \frac{A^2}{4}(M(f+f_c-f_c-W)u_{-1}(-f+f_c+W-f_c) \\
&\quad + M(f+f_c-f_c+W)u_{-1}(f+f_c+W-f_c) \\
&\quad + M(f-2f_c-W)u_{-1}(f-2f_c-W) \\
&\quad + M(f+2f_c+W)u_{-1}(-f-2f_c-W)) \\
&= \frac{A^2}{4}(M(f-W)u_{-1}(-f+W) + M(f+W)u_{-1}(f+W) \\
&\quad + M(f-2f_c-W)u_{-1}(f-2f_c-W) + M(f+2f_c+W)u_{-1}(-f-2f_c-W))
\end{aligned}$$

The LPF will cut-off the double frequency components, leaving the spectrum

$$Y(f) = \frac{A^2}{4}[M(f-W)u_{-1}(-f+W) + M(f+W)u_{-1}(f+W)]$$

The next figure depicts $Y(f)$ for $M(f)$ as shown in Fig. P-5.12.



2) As it is observed from the spectrum $Y(f)$, the system shifts the positive frequency components to the negative frequency axis and the negative frequency components to the positive frequency axis. If we transmit the signal $y(t)$ through the system, then we will get a scaled version of the original spectrum $M(f)$.

Note: $U_1(x)$ is a unit step function:

$$u_{-1}(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

Problem [redacted] 8

The input to the upper LPF is

$$\begin{aligned} u_u(t) &= \cos(2\pi f_m t)\cos(2\pi f_1 t) \\ &= \frac{1}{2}\left[\cos(2\pi(f_1 - f_m)t) + \cos(2\pi(f_1 + f_m)t)\right] \end{aligned}$$

whereas the input to the lower LPF is

$$\begin{aligned} u_l(t) &= \cos(2\pi f_m t)\sin(2\pi f_1 t) \\ &= \frac{1}{2}\left[\sin(2\pi(f_1 - f_m)t) + \sin(2\pi(f_1 + f_m)t)\right] \end{aligned}$$

If we select f_1 such that $|f_1 - f_m| < W$ and $f_1 + f_m > W$, then the two lowpass filters will cut-off the frequency components outside the interval $[-W, W]$, so that the output of the upper and lower LPF is

$$\begin{aligned} y_u(t) &= \cos(2\pi(f_1 - f_m)t) \\ y_l(t) &= \sin(2\pi(f_1 - f_m)t) \end{aligned}$$

The output of the Weaver's modulator is

$$u(t) = \cos(2\pi(f_1 - f_m)t)\cos(2\pi f_2 t) - \sin(2\pi(f_1 - f_m)t)\sin(2\pi f_2 t)$$

which has the form of a SSB signal since $\sin(2\pi(f_1 - f_m)t)$ is the Hilbert transform of $\cos(2\pi(f_1 - f_m)t)$. If we write $u(t)$ as

$$u(t) = \cos(2\pi(f_1 + f_2 - f_m)t)$$

then with $f_1 + f_2 - f_m = f_c + f_m$ we obtain an USSB signal centered at f_c, whereas with $f_1 + f_2 - f_m = f_c - f_m$ we obtain the LSSB signal. In both cases the choice of f_c and f_1 uniquely determine f_2.