ELG3175
Introduction to Communication Systems

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Solutions

Assignment #4

Problem [redacted] #1

1)

$$
\begin{aligned}
u(t) &= 5\cos(1800\pi t) + 20\cos(2000\pi t) + 5\cos(2200\pi t) \\
&= 20\left(1 + \frac{1}{2}\cos(200\pi t)\right)\cos(2000\pi t)
\end{aligned}
$$

The modulating signal is $m(t) = \cos(2\pi 100t)$ whereas the carrier signal is $c(t) = 20\cos(2\pi 1000t)$.

2) Since $-1 \leq \cos(2\pi 100t) \leq 1$, we immediately have that the modulation index is $a = \frac{1}{2}$.

3) The power of the carrier component is $P_{\text{carrier}} = \frac{400}{2} = 200$, whereas the power in the sidebands is $P_{\text{sidebands}} = \frac{400a^2}{2} \cdot \frac{1}{2} = \cancel{50}$ 25. Hence,

$$
\frac{P_{\text{sidebands}}}{P_{\text{carrier}}} = \frac{\cancel{50}\ 25}{200} = \frac{1}{8}
$$

!

#2 Using $\sin^2 \omega t = \frac{1}{2}(1 - \cos 2\omega t)$ and

$\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$, we obtain:

$$\sin^3 \omega t = \sin\omega t \cdot \frac{1}{2}(1 - \cos 2\omega t) = \frac{1}{4}(3\sin\omega t - \sin 3\omega t)$$

HT is $-\pi/2$ shift in frequency domain, i.e.

$$\widehat{\sin\omega t} = -\cos\omega t$$

Hence,

$$\widehat{\sin^3 \omega t} = \frac{1}{4}(\cos 3\omega t - 3\cos\omega t)$$

Note that

$$\cos^3 \omega t = \cos\omega t \cdot \frac{1}{2}(1 + \cos 2\omega t) = \frac{1}{4}(3\cos\omega t + \cos 3\omega t)$$

(we used here $\cos^2 \omega t = \frac{1}{2}(1 + \cos 2\omega t)$ and

$\cos\alpha \cos\beta = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$). Hence,

$\widehat{\sin^3 \omega t} \neq -\cos^3 \omega t$. This it true simply because

$-\pi/2$ phase shift applies to sinusoidal singnals only, while $\sin^3 \omega t$ is not a sinusoidal signal (contains several sinusoidal signals of different frequencies).

Problem #3

If $x(t)$ is even then $X(f)$ is a real and even function and therefore $-j\,\mathrm{sgn}(f)X(f)$ is an imaginary and odd function. Hence, its inverse Fourier transform $\hat{x}(t)$ will be odd. If $x(t)$ is odd then $X(f)$ is imaginary and odd and $-j\,\mathrm{sgn}(f)X(f)$ is real and even and, therefore, $\hat{x}(t)$ is even.

#4 $\quad x(t) \Rightarrow$ lowpass, $\quad y(t) = x(t) \cdot \cos \omega_0 t$

Spectrum of $y(t)$: $\quad S_y(\omega) = \frac{1}{2\pi} S_x(\omega) * \left(\frac{4\pi}{4} \delta(\omega - \omega_0) + \right.$

$\left. + \pi \delta(\omega + \omega_0) \right) = \dfrac{\pi S_x(\omega - \omega_0) + \pi S_x(\omega - \omega_0)}{2\pi} =$

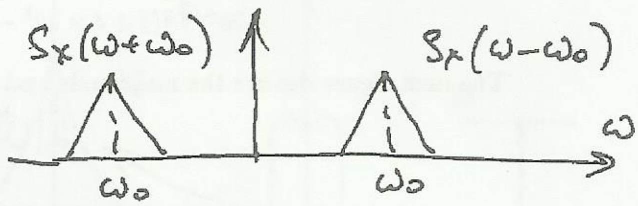
$= \frac{1}{2} S_x(\omega - \omega_0) + \frac{1}{2} S_x(\omega + \omega_0)$

Since $f_0 \gg B$, where B is bandwith of $x(t)$,

$S_x(\omega - \omega_0)$ is non-zero for positive frequencies only (around ω_0) and $S_x(\omega + \omega_0)$ is non-zero for negative frequencies only, see picture.

Hence,

$S_{\hat{y}}(\omega) = -\frac{j}{2} S_x(\omega - \omega_0) +$
$+ \frac{j}{2} S_x(\omega + \omega_0)$.



positive freq. $\quad$ negative freq.

$S_{\hat{y}}(\omega) = S_x(\omega) * \left[-\frac{j}{2} \delta(\omega - \omega_0) + \frac{j}{2} \delta(\omega + \omega_0) \right] \Rightarrow$

$\Rightarrow \hat{y}(t) = x(t) \cdot \widehat{\cos \omega_0 t} = x(t) \cdot \sin \omega t$

Problem #5

$f_c = 4\times10^3$ $f_{m1} = \frac{200}{\pi}$ $f_{m2} = \frac{50}{\pi}$

The modulated signal is

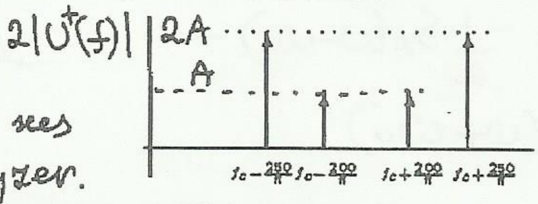
$$
\begin{aligned}
u(t) &= m(t)c(t) = Am(t)\cos(2\pi 4\times 10^3 t)\\
&= A\left[2\cos(2\pi\frac{200}{\pi}t) + 4\sin(2\pi\frac{250}{\pi}t+\frac{\pi}{3})\right]\cos(2\pi 4\times 10^3 t)\\
&= A\cos(2\pi(4\times10^3+\frac{200}{\pi})t) + A\cos(2\pi(4\times10^3-\frac{200}{\pi})t)\\
&\quad +2A\sin(2\pi(4\times10^3+\frac{250}{\pi})t+\frac{\pi}{3}) - 2A\sin(2\pi(4\times10^3-\frac{250}{\pi})t-\frac{\pi}{3})
\end{aligned}
$$

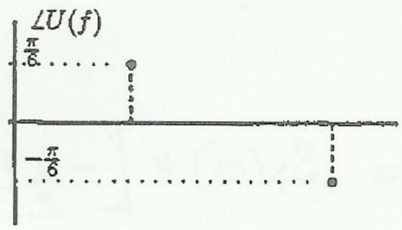
Taking the Fourier transform of the previous relation, we obtain

$$
\begin{aligned}
U(f) &= A\left[\delta(f-\frac{200}{\pi})+\delta(f+\frac{200}{\pi})+\frac{2}{j}e^{j\frac{\pi}{3}}\delta(f-\frac{250}{\pi})-\frac{2}{j}e^{-j\frac{\pi}{3}}\delta(f+\frac{250}{\pi})\right]\\
&\quad \star\frac{1}{2}[\delta(f-4\times10^3)+\delta(f+4\times10^3)]\\
&= \frac{A}{2}\left[\delta(f-\underbrace{4\times10^3}_{f_c}-\frac{200}{\pi})+\delta(f-4\times10^3+\frac{200}{\pi})\right.\\
&\quad +2e^{-j\frac{\pi}{6}}\delta(f-4\times10^3-\frac{250}{\pi})+2e^{j\frac{\pi}{6}}\delta(f-4\times10^3+\frac{250}{\pi})\\
&\quad +\delta(f+4\times10^3-\frac{200}{\pi})+\delta(f+4\times10^3+\frac{200}{\pi})\\
&\quad \left.+2e^{-j\frac{\pi}{6}}\delta(f+4\times10^3-\frac{250}{\pi})+2e^{j\frac{\pi}{6}}\delta(f+4\times10^3+\frac{250}{\pi})\right]
\end{aligned}
$$

The next figure depicts the magnitude and the phase of the spectrum as seen on spectrum analyzer

this is what one sees on a spectrum analyzer.





To find the power content of the modulated signal we write $u^2(t)$ as

$$
\begin{aligned}
u^2(t) &= A^2\cos^2(2\pi(4\times10^3+\frac{200}{\pi})t) + A^2\cos^2(2\pi(4\times10^3-\frac{200}{\pi})t)\\
&\quad +4A^2\sin^2(2\pi(4\times10^3+\frac{250}{\pi})t+\frac{\pi}{3}) + 4A^2\sin^2(2\pi(4\times10^3-\frac{250}{\pi})t-\frac{\pi}{3})\\
&\quad +\text{terms of cosine and sine functions in the first power}
\end{aligned}
$$

Hence,

$$
P = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} u^2(t)dt = \frac{A^2}{2}+\frac{A^2}{2}+\frac{4A^2}{2}+\frac{4A^2}{2} = 5A^2
$$

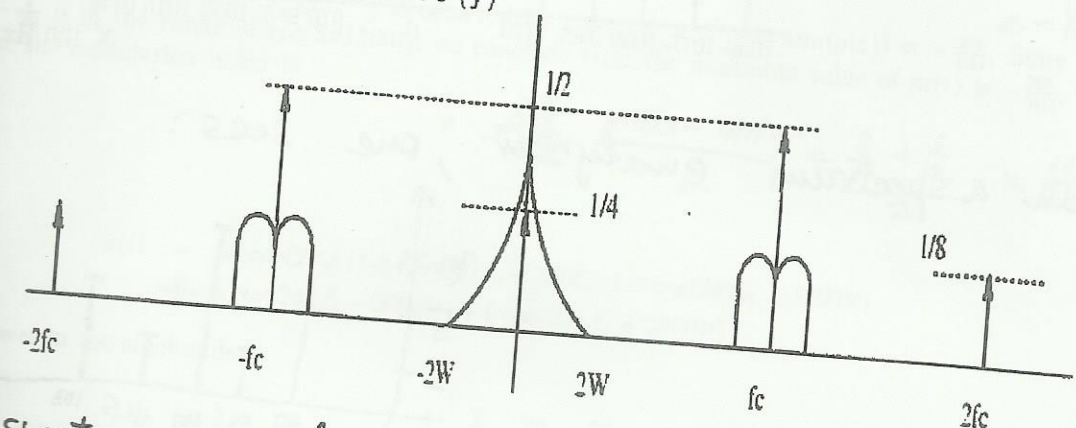
Problem #6

$$
\begin{aligned}
y(t) &= x(t) + \frac{1}{2}x^2(t) \\
&= m(t) + \cos(2\pi f_c t) + \frac{1}{2}\left(m^2(t) + \cos^2(2\pi f_c t) + 2m(t)\cos(2\pi f_c t)\right) \\
&= m(t) + \cos(2\pi f_c t) + \frac{1}{2}m^2(t) + \frac{1}{4} + \frac{1}{4}\cos(2\pi 2 f_c t) + m(t)\cos(2\pi f_c t)
\end{aligned}
$$

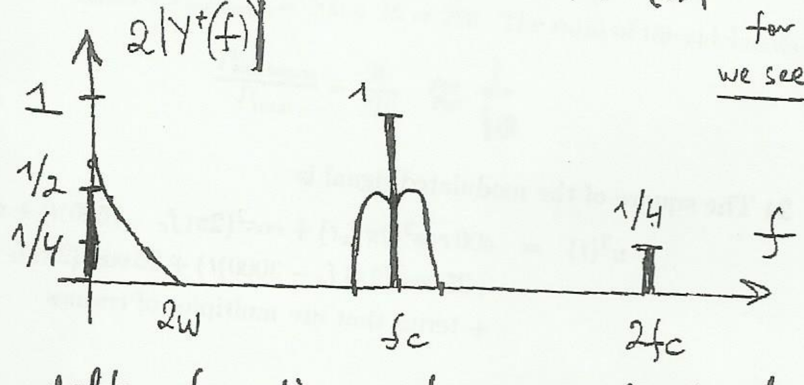
Taking the Fourier transform of the previous, we obtain

$$
\begin{aligned}
Y(f) &= M(f) + \frac{1}{2}M(f) \star M(f) + \frac{1}{2}\left(M(f-f_c) + M(f+f_c)\right) \\
&\quad + \frac{1}{4}\delta(f) + \frac{1}{2}\left(\delta(f-f_c) + \delta(f+f_c)\right) + \frac{1}{8}\left(\delta(f-2f_c) + \delta(f+2f_c)\right)
\end{aligned}
$$

The next figure depicts the spectrum $Y(f)$



One the spectrum analyzer, we see $2|Y^{+}(f)|$ → except for $f=0$: we see $Y(0)$!



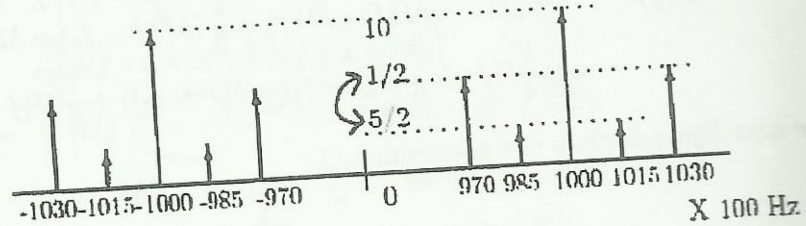
All "heights" of delta-functions have a finite height on a spectrum analyzer, equal to the magnitude of appropriate sinusoid in the spectrum.

Problem #7

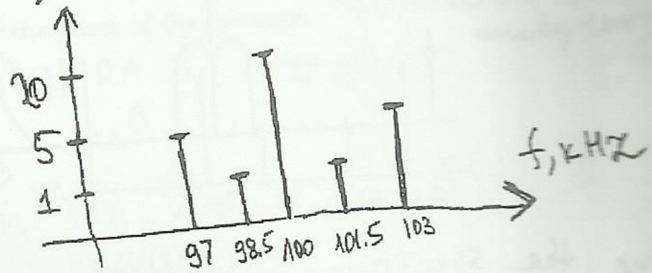
1) The spectrum of $u(t)$ is

$$
\begin{aligned}
U(f) &= \frac{20}{2}[\delta(f-f_c)+\delta(f+f_c)] \\
&+\frac{2}{4}[\delta(f-f_c-1500)+\delta(f-f_c+1500) \\
&+\delta(f+f_c-1500)+\delta(f+f_c+1500)] \\
&+\frac{10}{4}[\delta(f-f_c-3000)+\delta(f-f_c+3000) \\
&+\delta(f+f_c-3000)+\delta(f+f_c+3000)]
\end{aligned}
$$

The next figure depicts the spectrum of $u(t)$.



On a spectrum analyzer, one sees:



2) The square of the modulated signal is

$$
\begin{aligned}
u^2(t) &= 400\cos^2(2\pi f_c t)+\cos^2(2\pi(f_c-1500)t)+\cos^2(2\pi(f_c+1500)t) \\
&+25\cos^2(2\pi(f_c-3000)t)+25\cos^2(2\pi(f_c+3000)t) \\
&+ \text{terms that are multiples of cosines}
\end{aligned}
$$

#7 (cont.)

If we integrate $u^2(t)$ from $-\frac{T}{2}$ to $\frac{T}{2}$, normalize the integral by $\frac{1}{T}$ and take the limit as $T \to \infty$, then all the terms involving cosines tend to zero, whereas the squares of the cosines give a value of $\frac{1}{2}$. Hence, the power content at the frequency $f_c = 10^5$ Hz is $P_{f_c} = \frac{400}{2} = 200$, the power content at the frequency P_{f_c+1500} is the same as the power content at the frequency P_{f_c-1500} and equal to $\frac{1}{2}$, whereas $P_{f_c+3000} = P_{f_c-3000} = \frac{25}{2}$.

3)

$$\begin{aligned} u(t) &= (20 + 2\cos(2\pi 1500t) + 10\cos(2\pi 3000t))\cos(2\pi f_c t) \\ &= 20(1 + \frac{1}{10}\cos(2\pi 1500t) + \frac{1}{2}\cos(2\pi 3000t))\cos(2\pi f_c t) \end{aligned}$$

This is the form of a conventional AM signal with message signal

$$\begin{aligned} m(t) &= \frac{1}{10}\cos(2\pi 1500t) + \frac{1}{2}\cos(2\pi 3000t) \\ &= \cos^2(2\pi 1500t) + \frac{1}{10}\cos(2\pi 1500t) - \frac{1}{2} \end{aligned}$$

$\min[m(t)] = -\frac{1}{10} - \frac{1}{2}$

$\max[m(t)] = \frac{1}{10} + \frac{1}{2} = \frac{3}{5}$;

$\approx -1/2$

The minimum of $g(z) = z^2 + \frac{1}{10}z - \frac{1}{2}$ is achieved for $z = -\frac{1}{20}$ and it is $\min(g(z)) = -\frac{201}{400}$. Since $z = -\frac{1}{20}$ is in the range of $\cos(2\pi 1500t)$, we conclude that the minimum value of $m(t)$ is $-\frac{201}{400}$. $\approx -\frac{1}{2}$

Hence, the modulation index is

$$a = \cancel{\frac{201}{400}} \quad \frac{\max - \min}{2} = \frac{\frac{3}{5} + \frac{1}{2}}{2} = \frac{11}{20} \approx \frac{1}{2}$$

4)

$$\begin{aligned} u(t) &= 20\cos(2\pi f_c t) + \cos(2\pi(f_c - 1500)t) + \cos(2\pi(f_c - 1500)t) \\ &+ 5\cos(2\pi(f_c - 3000)t) + 5\cos(2\pi(f_c + 3000)t) \end{aligned}$$

The power in the sidebands is

$$P_{\text{sidebands}} = \frac{1}{2} + \frac{1}{2} + \frac{25}{2} + \frac{25}{2} = 26$$

The total power is $P_{\text{total}} = P_{\text{carrier}} + P_{\text{sidebands}} = 200 + 26 = 226$. The ratio of the sidebands power to the total power is

$$\frac{P_{\text{sidebands}}}{P_{\text{total}}} = \frac{26}{226} \approx \frac{1}{10}$$

8

[redacted]

Using Rayleigh's theorem of the Fourier transform we have

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

and

$$E_{\hat{x}} = \int_{-\infty}^{\infty} |\hat{x}(t)|^2 dt = \int_{-\infty}^{\infty} |-j\mathrm{sgn}(f)X(f)|^2 df$$

Noting the fact that $|-j\mathrm{sgn}(f)|^2 = 1$ except for $f = 0$, and the fact that $X(f)$ does not contain any impulses at the origin we conclude that $E_x = E_{\hat{x}}$.

#9

Problem 2.43
We know that

$$\begin{aligned} x(t) &= x_c(t)\cos(2\pi f_0 t) - x_s(t)\sin(2\pi f_0 t) \\ \hat{x}(t) &= x_c(t)\sin(2\pi f_0 t) + x_s(t)\cos(2\pi f_0 t) \end{aligned}$$

We can write these relations in matrix notation as

$$\begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix} = \begin{pmatrix} \cos(2\pi f_0 t) & -\sin(2\pi f_0 t) \\ \sin(2\pi f_0 t) & \cos(2\pi f_0 t) \end{pmatrix} \begin{pmatrix} x_c(t) \\ x_s(t) \end{pmatrix} = R \begin{pmatrix} x_c(t) \\ x_s(t) \end{pmatrix}$$

The rotation matrix R is nonsingular ($\det(R) = 1$) and its inverse is

$$R^{-1} = \begin{pmatrix} \cos(2\pi f_0 t) & \sin(2\pi f_0 t) \\ -\sin(2\pi f_0 t) & \cos(2\pi f_0 t) \end{pmatrix}$$

Thus

$$\begin{pmatrix} x_c(t) \\ x_s(t) \end{pmatrix} = R^{-1} \begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix} = \begin{pmatrix} \cos(2\pi f_0 t) & \sin(2\pi f_0 t) \\ -\sin(2\pi f_0 t) & \cos(2\pi f_0 t) \end{pmatrix} \begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix}$$

and the result follows.