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Introduction to Communication Systems

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Solutions

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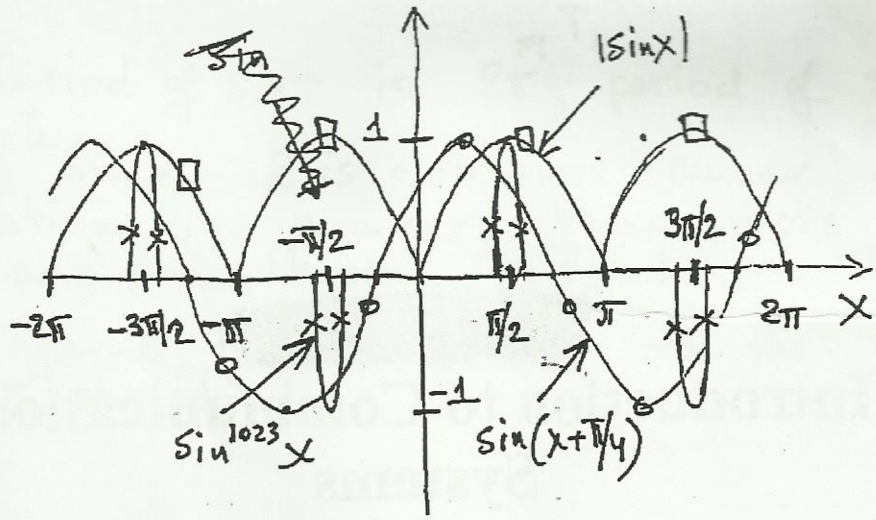

Since $(\sin x)^{1023}$ decreases very fast when x deviates from $\pi/2$ ($3\pi/2$, $-\pi/2$ etc.), it is very narrow lobe on the graph (remind that $|\sin x| \leq 1$).

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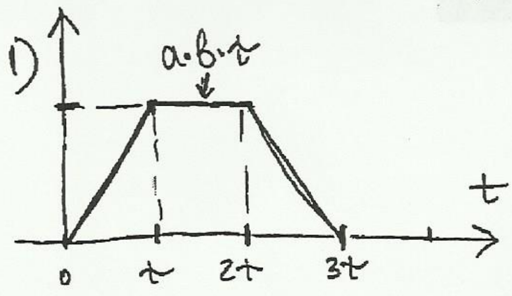
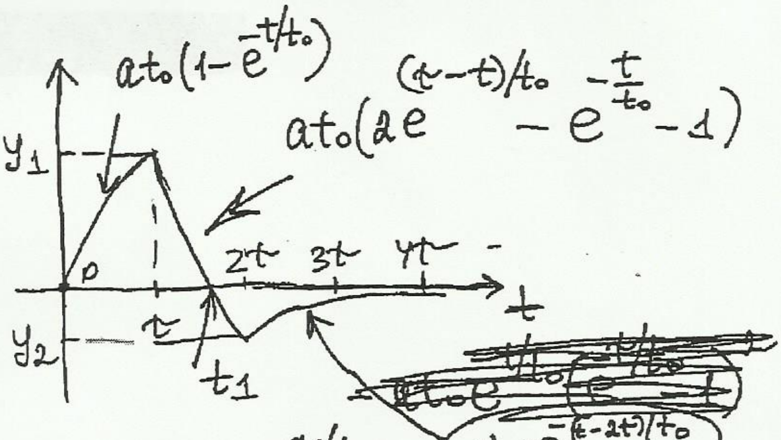
2 Using the definition of convolution,

$$z(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

we first invert $h(\tau)$, i.e. $h(\tau) \to h(-\tau)$ and then find convolution as an area under the product of $x(\tau)h(t-\tau)$.





$$\begin{cases} y_1 = at_0(1-e^{-\tau/t_0}) \\ y_2 = -at_0 (e^{-\tau/t_0} - 1)^2 \\ t_1 = t_0 \ln(2e^{\tau/t_0} - 1) \end{cases}$$

③ $\cos(\alpha+\beta) = Re\{e^{j(\alpha+\beta)}\} = Re\{e^{j\alpha}e^{j\beta}\} =$

$= Re\{e^{j\alpha}\}Re\{e^{j\beta}\} - Im\{e^{j\alpha}\}Im\{e^{j\beta}\} =$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Similar argument for $\sin(\alpha+\beta) = Im\{e^{j(\alpha+\beta)}$

noting that $Im[Z_1 Z_2] = Re\{Z_1\}Im\{Z_2\} + Im\{Z_1\}\cdot$

$\cdot Re\{Z_2\}$: $\sin(\alpha+\beta) = Re\{e^{j\alpha}\}Im\{e^{j\beta}\} + Im\{e^{j\alpha}\}\cdot$

$\cdot Re\{e^{j\beta}\} = \cos\alpha\sin\beta + \sin\alpha\cos\beta$.

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④ ⓐ system is linear if $L\{ax_1 + bx_2\} = aL\{x_1\} +$ $+ bL\{x_2\}$. In our case, $L\{ax_1\} = a^*L\{x_1\}$

Hence, it is not linear.

ⓑ Time invariance ~~if~~ $y(t) = L\{x(t)\} \Rightarrow$ $\Rightarrow y(t-t_0) = L\{x(t-t_0)\} \rightarrow$ yes, it is.

ⓒ ~~test~~ Casuality means that output dependes on preceeding ~~mom~~ values of input and on the present ~~fo~~ value, but not on future. In our case, output depends on input at the same moment of time. Hence, yes, it is casual.

ⓓ stability means that bounded input, $|x(t)| \le X_{max}$ produces bounded output, $|y(t)| \le y_{max}$. Yes, it is stable because $|x(t)| = |y(t)|$.

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(2) $x_2(t) = 1$. It follows then that $x_{2,0} = 1$ and $x_{2,n} = 0,\ \forall n \neq 0$.

(4) The signal $\cos(t)$ is periodic with period $T_1 = 2\pi$ whereas $\cos(2.5t)$ is periodic with period $T_2 = 0.8\pi$. It follows then that $\cos(t) + \cos(2.5t)$ is periodic with period $T = 4\pi$. The trigonometric Fourier series of the even signal $\cos(t) + \cos(2.5t)$ is

$$\begin{aligned}\cos(t) + \cos(2.5t) &= \sum_{n=1}^{\infty} \alpha_n \cos(2\pi \frac{n}{T_0} t) \\ &= \sum_{n=1}^{\infty} \alpha_n \cos(\frac{n}{2} t)\end{aligned}$$

By equating the coefficients of $\cos(\frac{n}{2}t)$ of both sides we observe that $a_n = 0$ for all n unless $n = 2, 5$ in which case $a_2 = a_5 = 1$. Hence $x_{4,2} = x_{4,5} = \frac{1}{2}$ and $x_{4,n} = 0$ for all other values of n.

(6) The signal $x_6(t)$ is periodic with period $T_0 = 2T$. We can write $x_6(t)$ as

$$\begin{aligned}x_6(t) &= \sum_{n=-\infty}^{\infty} \delta(t - n2T) - \sum_{n=-\infty}^{\infty} \delta(t - T - n2T) \\ &= \frac{1}{2T} \sum_{n=-\infty}^{\infty} e^{j\pi \frac{n}{T} t} - \frac{1}{2T} \sum_{n=-\infty}^{\infty} e^{j\pi \frac{n}{T}(t-T)} \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2T}(1 - e^{-j\pi n}) e^{j2\pi \frac{n}{2T} t}\end{aligned}$$

However, this is the Fourier series expansion of $x_6(t)$ and we identify $x_{6,n}$ as

$$x_{6,n} = \frac{1}{2T}(1 - e^{-j\pi n}) = \frac{1}{2T}(1 - (-1)^n) = \begin{cases} 0 & n \text{ even} \\ \frac{1}{T} & n \text{ odd} \end{cases}$$

Note: $\underline{x_n = C_n}$ (different notations!)

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d) The signal $x_8(t)$ is real even and periodic with period $T_0 = \frac{1}{2f_0}$. Hence, $x_{8,n} = a_{8,n}/2$ or

$$\begin{aligned}
x_{8,n} &= 2f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t)\cos(2\pi n 2 f_0 t)dt \\
&= f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+2n)t)dt + f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-2n)t)dt \\
&= \frac{1}{2\pi(1+2n)} \sin(2\pi f_0(1+2n)t)\Big|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-2n)} \sin(2\pi f_0(1-2n)t)\Big|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\
&= \frac{(-1)^n}{\pi}\left[\frac{1}{(1+2n)} + \frac{1}{(1-2n)}\right]
\end{aligned}$$

Note different notations: $x_n = c_n$

#6

Problem 6

a) The signal $y(t) = x(t-t_0)$ is periodic with period $T = T_0$.

$$\begin{aligned} y_n &= \frac{1}{T_0}\int_\alpha^{\alpha+T_0} x(t-t_0)e^{-j2\pi\frac{n}{T_0}t}dt \\ &= \frac{1}{T_0}\int_{\alpha-t_0}^{\alpha-t_0+T_0} x(v)e^{-j2\pi\frac{n}{T_0}}(v+t_0)dv \\ &= e^{-j2\pi\frac{n}{T_0}t_0}\frac{1}{T_0}\int_{\alpha-t_0}^{\alpha-t_0+T_0} x(v)e^{-j2\pi\frac{n}{T_0}v}dv \\ &= x_n e^{-j2\pi\frac{n}{T_0}t_0} \end{aligned}$$

b) For $y(t)$ to be periodic there must exist T such that $y(t+mT) = y(t)$. But $y(t+T) = x(t+T)e^{j2\pi f_0 t}e^{j2\pi f_0 T}$ so that $y(t)$ is periodic if $T = T_0$ (the period of $x(t)$) and $f_0 T = k$ for some k in $\mathcal{Z}$. In this case

$$\begin{aligned} y_n &= \frac{1}{T_0}\int_\alpha^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}e^{j2\pi f_0 t}dt \\ &= \frac{1}{T_0}\int_\alpha^{\alpha+T_0} x(t)e^{-j2\pi\frac{(n-k)}{T_0}t}dt = x_{n-k} \end{aligned}$$

c) The signal $y(t)$ is periodic with period $T = T_0/\alpha$.

$$\begin{aligned} y_n &= \frac{1}{T}\int_\beta^{\beta+T} y(t)e^{-j2\pi\frac{n}{T}t}dt = \frac{\alpha}{T_0}\int_\beta^{\beta+\frac{T_0}{\alpha}} x(\alpha t)e^{-j2\pi\frac{n\alpha}{T_0}t}dt \\ &= \frac{1}{T_0}\int_{\beta\alpha}^{\beta\alpha+T_0} x(v)e^{-j2\pi\frac{n}{T_0}v}dv = x_n \end{aligned}$$

where we used the change of variables $v = \alpha t$. → note that fundamental frequenc[y] is different $\omega_0' = \alpha\omega_0$!

d)

$$\begin{aligned} y_n &= \frac{1}{T_0}\int_\alpha^{\alpha+T_0} x'(t)e^{-j2\pi\frac{n}{T_0}t}dt \\ &= \frac{1}{T_0}x(t)e^{-j2\pi\frac{n}{T_0}t}\Big|_\alpha^{\alpha+T_0} - \frac{1}{T_0}\int_\alpha^{\alpha+T_0}(-j2\pi\frac{n}{T_0})e^{-j2\pi\frac{n}{T_0}t}dt \\ &= j2\pi\frac{n}{T_0}\frac{1}{T_0}\int_\alpha^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}dt = j2\pi\frac{n}{T_0}x_n = j\omega_0 n\cdot C_n \end{aligned}$$

Note: different notations: $\underline{X_n = C_n}$

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Problem [redacted]7

Using the result [redacted]

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)x^*(t)dt = \sum_{n=-\infty}^{\infty} |x_n|^2$$

Since the signal has finite power

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} |x(t)|^2 dt = K < \infty$$

Thus, $\sum_{n=-\infty}^{\infty} |x_n|^2 = K < \infty$. The last implies that $|x_n| \to 0$ as $n \to \infty$. To see this write

$$\sum_{n=-\infty}^{\infty} |x_n|^2 = \sum_{n=-\infty}^{-M} |x_n|^2 + \sum_{n=-M}^{M} |x_n|^2 + \sum_{n=M}^{\infty} |x_n|^2$$

Each of the previous terms is positive and bounded by K. Assume that $|x_n|^2$ does not converge to zero as n goes to infinity and choose $\epsilon > 0$. Then there exists a subsequence of x_n, x_{n_k}, such that

$$|x_{n_k}| > \epsilon > 0, \qquad \text{for } n_k > N \geq M$$

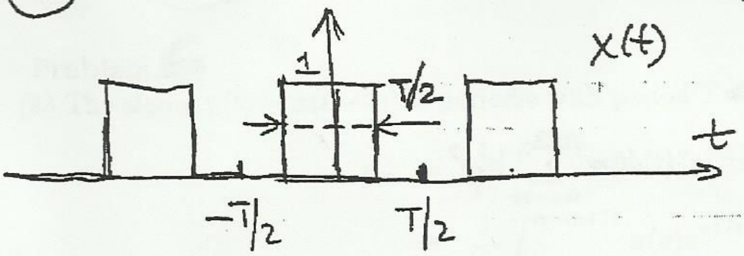
Then

$$\sum_{n=M}^{\infty} |x_n|^2 \geq \sum_{n=N}^{\infty} |x_n|^2 \geq \sum_{n_k} |x_{n_k}|^2 = \infty$$

since $\sum_{n_k} |X_{n_k}|^2 > \infty \cdot \mathcal{E} = \infty$.

This contradicts our assumption that $\sum_{n=M}^{\infty} |x_n|^2$ is finite. Thus $|x_n|$, and consequently x_n, should converge to zero as $n \to \infty$.

⑧ Consider the following pulse train:



Its spectrum is

$$|C_n| = \begin{cases} \frac{1}{n\pi}, & n = 2k+1 \\ 0, & n = 2k \\ 1/2, & n = 0 \end{cases}$$

$$|C_n| = |C_{-n}|$$

Its power is

$$P_x = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T}\int_{-T/4}^{T/4} dt = \frac{1}{2};$$

Using Parseval theorem, $P_x = \sum_{n=-\infty}^{+\infty} |C_n|^2 = 2\sum_{n=0}^{\infty} |C_n|$

$$-\frac{1}{4} = \cancel{2\sum_{k=0}^{\infty}\frac{1}{n^2\pi^2}} = \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} + \frac{1}{4} = \frac{1}{2} \Rightarrow$$

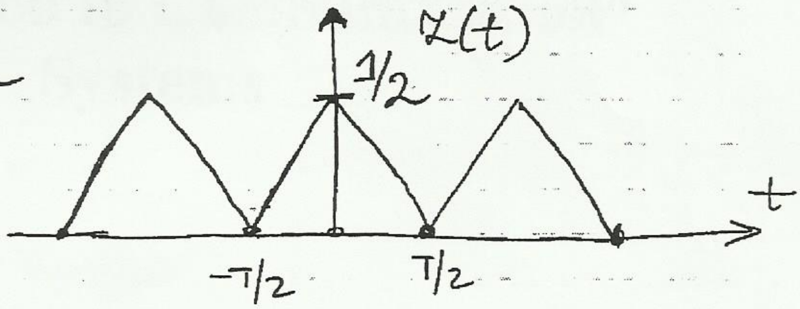
$$\Rightarrow \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8};$$

(9) Consider the convolution:

$$z(t) = \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)\, x(t-\tau)\, d\tau$$

where $x(t)$ is the pulse train in prob. #9.

Using the convolution theorem the spectrum of $z(t)$ is



$$|C_n'| = |C_n|^2 = \begin{cases} \frac{1}{\pi^2 n^2}, & n = 2k+1 \\ 0, & n = 2k \\ 1/4, & n = 0 \end{cases}$$

Its power is: $P_z = \frac{1}{T}\int_{-T/2}^{T/2} |z(t)|^2 dt = \frac{2}{T}\int_0^{T/2}\left(\frac{1}{2}-\frac{t}{T}\right)^2 d$

$= \frac{1}{12}$; On the other hand:

$$P_z = \sum_{n=-\infty}^{\infty} |C_n'|^2 = \frac{2}{\pi^4}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} + \frac{1}{16} = \frac{1}{12} \Rightarrow$$

$$\Rightarrow \sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{\pi^4}{96};$$

(10) Period of $\sin t$ is $2\pi \Rightarrow T_1$, period of $\sin(\pi t)$ is $2. \Rightarrow T_2$ Since 2 is a rational numer and 2π is irrational number, there exists no such n and k that $nT_1 = kT_2$. Hence, the joint period is $\infty \rightarrow$ the signal is not periodic!