Solutions –
Problem Set #3

Question #1 – We must show R is an equiv. relation, hence must show it is reflexive, symmetric, transitive.

- **Reflexive** $n^2-n^2=0$, which is divisible by 3. So nRn for all $n\in\mathbb{Z}$
- **Symmetric** Suppose nRm. Thus n^2-m^2 is divisible by 3. We must show mRn, i.e. m^2-n^2 is divisible by 3. Note $m^2-n^2=-(n^2-m^2)$. So m^2-n^2 is divisible by 3
- **Transitive** Suppose nRm & mRp. So n^2-m^2 is divisible by 3 and m^2-p^2 is divisible by 3. So $(n^2-m^2)+(m^2-p^2)=n^2-p^2$ is divisible by 3, since the sum of 2 #'s divisible by 3 is divisible by 3.

Now note n is in the equivalence class of 1 if $n^2-1^2=n^2-1$ is divisible by 3. So 1, 2, 4, 5 are in this class

Question #2 These graphs are not isomorphic. The first has 6 cycles of length 4, and the second only has 2.

Question #3 Proof by Induction

Base Case $n=1$. In this case $2^{2n}-1=3$, which is divisible by 3.

Induction Hypothesis We assume that for some fixed n, $2^{2n}-1$ is divisible by 3.

Induction Step We must prove $2^{2(n+1)}-1$ is divisible by 3. Note

$$2^{2(n+1)}-1 = 2^{2n+2}-1 = 4(2^n)-1 =$$
$$4(2^n)-4+4-1 = 4(2^n-1)+3$$

Since 2^n-1 is divisible by 3, so is $4(2^n-1)$. Thus so is $4(2^n-1)+3$. □

Q#4 a) Let x = # of vertices of degree 4. We have

$$2|E| = 42 = \sum_{v\in V} \deg(v) = 7+6+21+4x$$

So $8=4x$ or $x=2$

b) We have $2|E| = \sum_{v\in V} \deg(v) = k|V|$

If k is odd, it is relatively prime to 2, so we must have k divides $|E|$.

Question 5

Part 1 We have $2|E| = \sum_{v \in V} deg(v) =$

$\sum_{v \in V} 3 = 3|V|$. So $2|E| = 4|V| - 20 = 3|V|$

So $|V| = 20$. Since $2|E| = 3|V|$, $|E| = 30$

Part 2 The most edges a graph can have occurs in K_n, which has $\frac{n(n-1)}{2}$ edges.

K_5 has 10 edges, K_6 has 15 edges K_7 has 21 edges.

So I need at least 7 vertices.

Question 6

Base Case $n=0$

$4(0)+1 = (0+1)(0+1)$ ✓

Induction Hypothesis

Assum $\sum_{i=0}^{n} (4i+1) = (n+1)(2n+1)$ (I H)

We must show

$$\sum_{i=0}^{n+1} (4i+1) = (n+2)(2n+3)$$

$$\sum_{i=0}^{n+1}(4i+1) = \sum_{i=0}^{n}(4i+1) + (4n+5) \overset{IH}{=} (n+1)(2n+1) + (4n+5)$$

$$= 2n^2+3n+1+4n+5 = 2n^2+7n+6 = (n+2)(2n+3) \checkmark$$

Q7 Was done in class

Q8 A graph G has 17 edges. Each vertex has degree at least 3, i.e. $\deg(v) \geq 3$.

So $2 \cdot 17 = \sum_{v \in V} \deg(v) \geq 3|V|$

So $34 \geq 3|V|$. So $|V|$ could be at most 11. $\square$