Lecture 7 - 1348

How to prove things

Statement of form $P \Rightarrow Q$

Direct Proof

Assume P, prove Q, based on P

Indirect Proof (Proof by con

Uses that $P \Rightarrow Q$ is equiv to $\neg Q \Rightarrow \neg P$

So assume $\neg Q$. Prove $\neg P$.

Ex: #18, p85 Suppose ~~n~~ n is an ~~a~~ integer. Prove:

If $3n+2$ is even, then n is even.

Proof: This can be proved directly. But let's do an indirect proof. Switching to $\neg Q \Rightarrow \neg P$, we get

If n is odd, then $3n+2$ is odd.

So we assume n is odd. So $n = 2m+1$, with m an integer.

Thus $3n+2 = 3(2m+1)+2 = 6m+5 =$

$2(3m+2)+1$.

So $3n+2$ is odd. $\square$

<u>Defn 2, p79</u> Let r be a real #. r is <u>rational</u>

if $r = \frac{p}{q}$, where p and q are integers. Note $q \neq 0$.

r is <u>irrational</u> if not rational.

<u>Fact</u> There are infinitely many irrational #'s.

$\left.\begin{array}{l}\pi \text{ is irrational} \\ e \text{ is irrational}\end{array}\right\}$ proofs are hard

$\sqrt{2}$ is irrational (later today)

#13 Prove that if x is irrational, then $1/x$ is irrational.

Method? Indirect $\neg Q \Rightarrow \neg P$ is

If $1/x$ is rational, then x is rational

Proof: Suppose $\frac{1}{x}$ is rational. So $1/x = m/n$. For m, n integers. Note $n \neq 0$, and $m \neq 0$, since $1/x$ cannot be 0.

Now $x = \frac{1}{1/x} = \frac{1}{m/n} = \frac{n}{m}$. This is well-defined, since $m \neq 0$. So x is rational. $\square$

Proof by Contradiction

We wish to prove the statement P. Suppose I can find a proof of

$$\neg P \Rightarrow Q$$

, where Q is a contradiction, i.e. a formula that is never true.

An example of a contradiction

$$r \wedge \neg r$$

$$\neg(r \Rightarrow r)$$

etc

If one can show $\neg P \Rightarrow Q$ where Q is a contradiction, then $\neg P$ must be false. So $\neg\neg P$ is true, and $\neg\neg P$ is equiv to P.

Thm: $\sqrt{2}$ is irrational.

Proof: Proof by contradiction.

We are trying to prove $P =$ "$\sqrt{2}$ is irrational"

Since a proof by contradiction, we assume ~~the~~ $\neg P$.
I.e. we assume $\sqrt{2}$ is rational

So $\sqrt{2} = \frac{a}{b}$, with a, b integers and a, b have no common factors, i.e. the fraction is reduced.

Since $\sqrt{2} = \frac{a}{b}$, we get $2 = \frac{a^2}{b^2}$ squaring both sides

So $2b^2 = a^2$. So a^2 is even.

Note that if a was odd, then a^2 would be odd.
(Prove this yourselves). So a is even.

So $a = 2c$, for some integer c. So
$$2b^2 = 4c^2$$
So $b^2 = 2c^2$. So b is even, as before.
We have a contradiction. Where? $\square$

Thm: (Euclid, 300 BC) There are infinitely many prime #'s.

Proof: It's a proof by contradiction.
So suppose the opposite:
$\neg P$ = There are finitely many primes

List them

$p_1, p_2, \ldots, p_n$

This is a complete list.

Consider the #

$$N = p_1 \cdot p_2 \cdot \cdots \cdot p_n + 1$$

Every # can be written as a product of prime #'s.

Where is the contradiction? $\square$

End Ch1. We are skipping 1.7

Set Notation

$\{a, b, c\}$

$\{x \mid x \geq 3\}$

<u>Ch 2</u> Def'n 1,2,3,4

A <u>set</u> is a collection of objects.

An object in a set is called an <u>element</u>

(Write $x \in X$, if the object x is an element of the set X.)

Two sets X and Y are <u>equal</u> if they have the same elements, i.e.

X and Y are equal if ~~&~~ and only if

$\forall x \, (x \in A \Leftrightarrow x \in B)$

A set A is a <u>subset</u> of B if every element of A is also an element of B.

Write $A \subseteq B$.

A is a <u>proper subset</u> of B if $A \subseteq B$ and $A \neq B$. This means there is an element of B that is not in A.

Thm: For any set S,

$$\emptyset \subseteq S$$

$$S \subseteq S$$

<u>Note</u> $\emptyset$ is the empty set. The set with no elements.

<u>Note</u> In defn of set, the word collection implies 2 things

1) There is no order. So

$$\{a,b,c\} = \{c,b,a\}$$

2) An element can only be in the set once. So $\{a,a,b,c\} = \{a,b,c\}$

finiteness

power set

cartesian

product