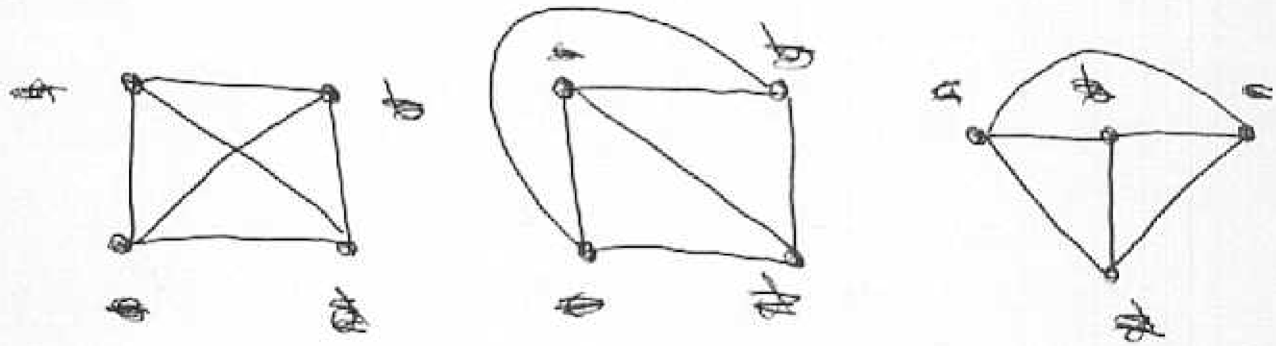
Lecture 21 - 1348

Last Time 9.1, 9.2

Today - New Topic - Graph Isomorphism

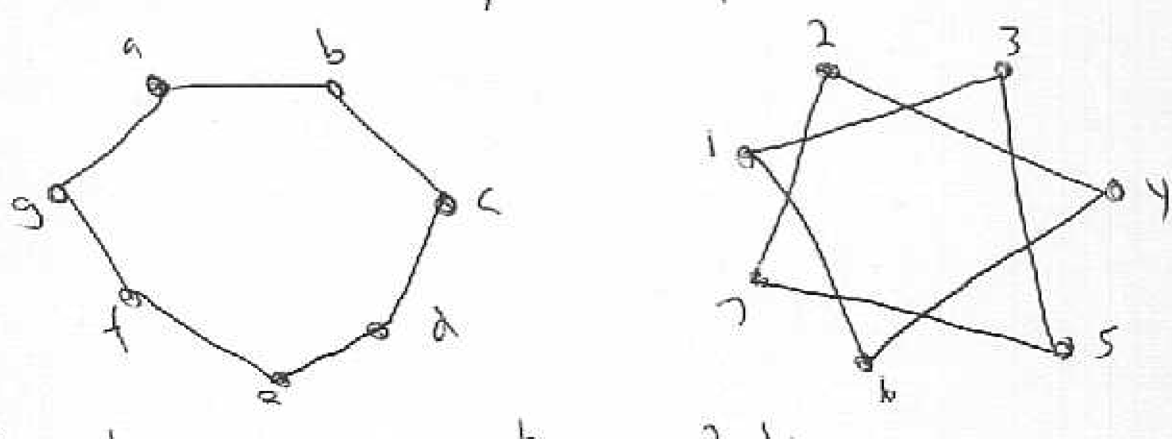
This will definitely be on final.

Idea Consider:



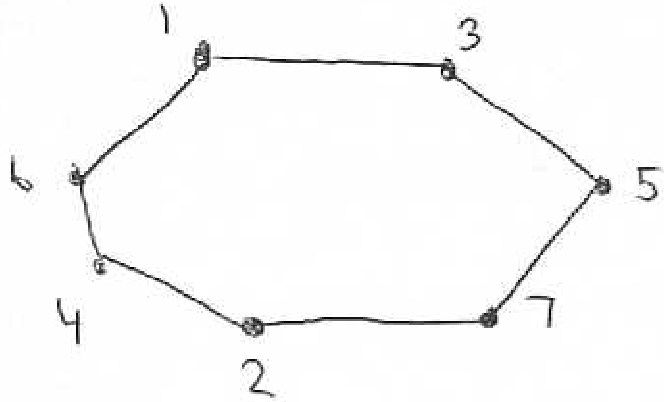
All 3 graphs are the same graph, just drawn differently.

More complicated example, #37 p 619



Are these two graphs the same? Yes

We will rearrange the positions of the second graph's vertices



Note the edges are exactly the same, just position of vertices is changed.

Defn, p615

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. We will assume loop-free, but this isn't important.

Then G_1 and G_2 are isomorphic if there is a bijective function

$$f: V_1 \to V_2 \quad \text{s.t.}$$

f is called an isomorphism

If $a, b \in V_1$, then

a is adjacent to b in G_1 if and only if $f(a)$ is adjacent to $f(b)$ in G_2

This is equivalent to saying

if a is adjacent to b in G_1, then $f(a)$ is adjacent to $f(b)$ in G_2

and

if a is not adjacent to b in G_1, then $f(a)$ is not adjacent to $f(b)$ in G_2

All our examples are isomorphic. For our last example, the bijection is

$$
\begin{aligned}
a &\longleftrightarrow 1\\
b &\longleftrightarrow 3\\
c &\longleftrightarrow 5\\
d &\longleftrightarrow 7\\
e &\longleftrightarrow 2\\
f &\longleftrightarrow 4\\
g &\longleftrightarrow 6
\end{aligned}
$$

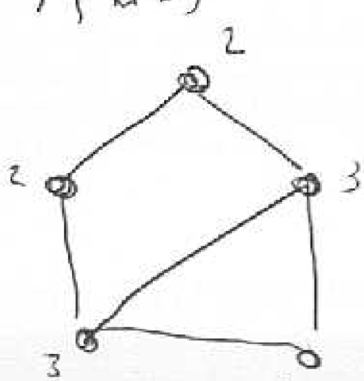
How do you show two graphs are not isomorphic?

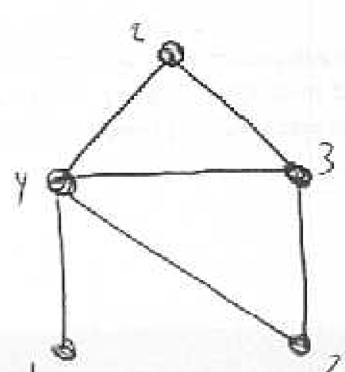
~~Note: There~~

Boring case -

If G_1 and G_2 have a different # of vertices or a different # of edges, they can not be isomorphic.

<u>Harder Example</u>: Are the following isomorphic (Ex 9, p66)





Here are several proofs they are not isomorphic

1) Count degrees

The first graph has 3 vertices of degree 2 and 2 of degree 3.

The second graph has 2 vertices of degree 2, 1 of degree 3, 1 of degree 4 and 1 of degree 1

This doesn't match, so the graphs are not isomorphic.

Fact An isomorphism takes a vertex of degree n to a vertex of degree n.

So, if G_1 and G_2 are iso, they must have same # of vertices of degree 0
" " " " " " 1
" " " " " " 2

etc.

2) Count Cycles

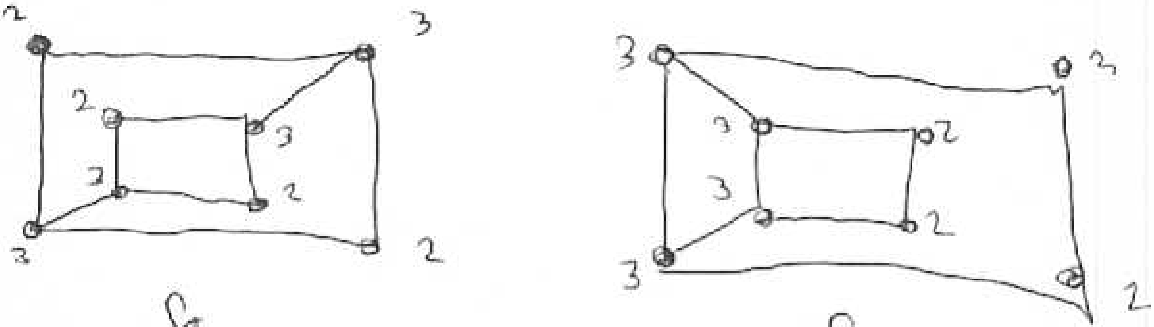
The first graph has 1 cycle of degree 3.
The second graph has 2 cycles of degree 3.

Hence they are not isomorphic

The first graph has 1 cycle of length 5,
the second has none

Hence they are not iso.

EX



G_1 $\qquad$ G_2

Both graphs have 8 vertices and 10 edges.

Both have 4 vertices of degree 3 and 4 of degree 2.

Note This does not imply they are iso.

In fact, they are not. G_2 has 3 cycles of length 4. G_1 has 2.

Also, G_2 has a cycle of length 4, all of whose vertices have degree 3. G_1 does not.

You are strongly advised to do exercises

34 - 44, p619

34 - Yes, they are iso. Easy

35 - Yes, they are iso

36 - No, first has a cycle of length 4

37 - Yes, already done

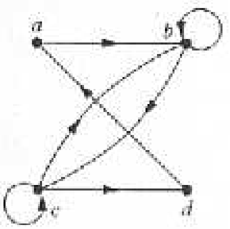
38 - Yes they are iso. Describe geometrically.

39 - Yes, they are iso

40 - No, second has 2 cycles of length 4, first has only 1

In Exercises 19–21 find the adjacency matrix of the given directed multigraph.

19.



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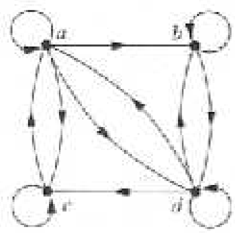

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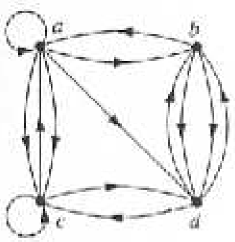


In Exercises 22–24 draw the graph represented by the given adjacency matrix.

22. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **23.** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$ **24.** $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

25. Is every zero–one square matrix that is symmetric and has zeros on the diagonal the adjacency matrix of a simple graph?

26. Use an incidence matrix to represent the graphs in Exercises 1 and 2.

27. Use an incidence matrix to represent the graphs in Exercises 13–15.

***28.** What is the sum of the entries in a row of the adjacency matrix for an undirected graph? For a directed graph?

***29.** What is the sum of the entries in a column of the adjacency matrix for an undirected graph? For a directed graph?

30. What is the sum of the entries in a row of the incidence matrix for an undirected graph?

31. What is the sum of the entries in a column of the incidence matrix for an undirected graph?

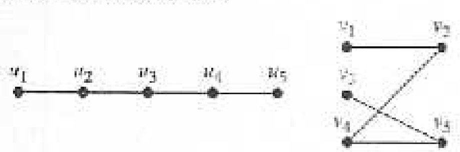
***32.** Find an adjacency matrix for each of these graphs.

a) K_n **b)** C_n **c)** W_n **d)** $K_{m,n}$ **e)** Q_n

***33.** Find incidence matrices for the graphs in parts (a)–(d) of Exercise 32.

In Exercises 34–44 determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

34.



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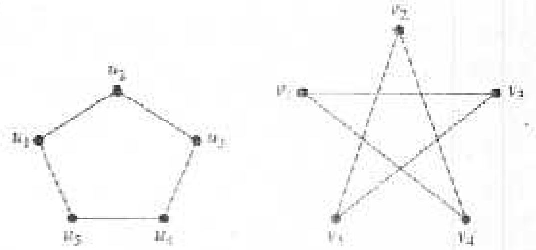

36.

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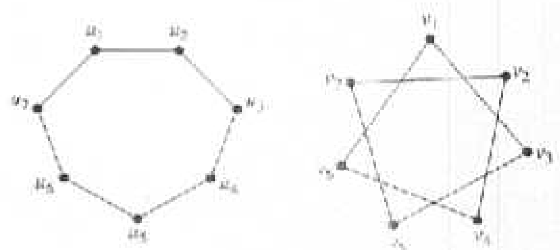

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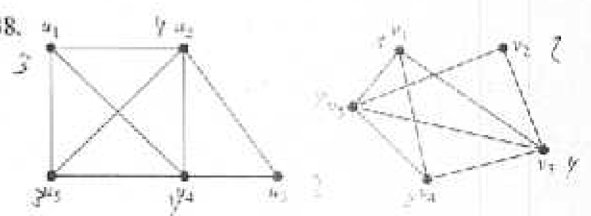

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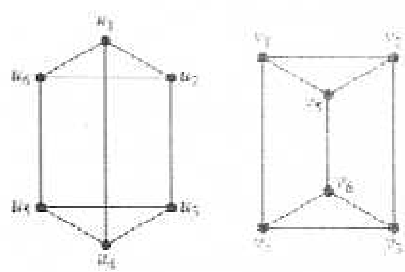

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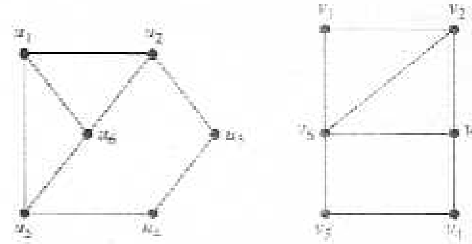

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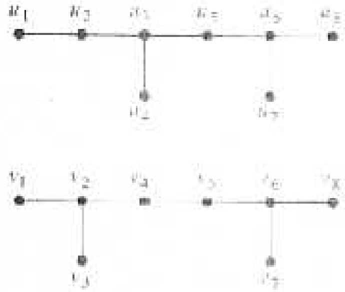

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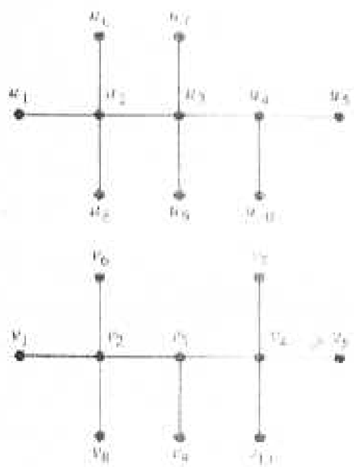

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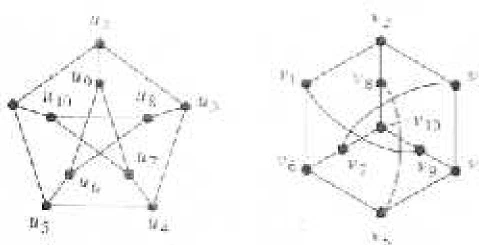

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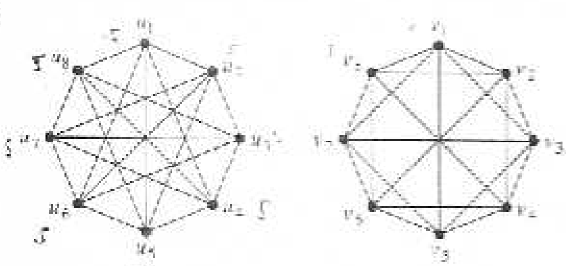


45. Show that isomorphism of simple graphs is an equivalence relation.

46. Suppose that G and H are isomorphic simple graphs. Show that their complementary graphs $\overline{G}$ and $\overline{H}$ are also isomorphic.

47. Describe the row and column of an adjacency matrix of a graph corresponding to an isolated vertex.

48. Describe the row of an incidence matrix of a graph corresponding to an isolated vertex.

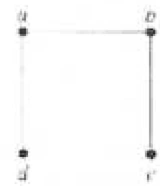
49. Show that the vertices of a bipartite graph with two or more vertices can be ordered so that its adjacency matrix has the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{bmatrix},$$

where the four entries shown are rectangular blocks.

A simple graph G is called **self-complementary** if G and $\overline{G}$ are isomorphic.

50. Show that this graph is self-complementary.



51. Find a self-complementary simple graph with five vertices.

***52.** Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.

53. For which integers n is C_n self-complementary?

54. How many nonisomorphic simple graphs are there with n vertices, when n is

a) 2? b) 3? c) 4?

55. How many nonisomorphic simple graphs are there with five vertices and three edges?

56. How many nonisomorphic simple graphs are there with six vertices and four edges?

57. Are the simple graphs with the following adjacency matrices isomorphic?

a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$