Lecture 18 - 1348

RB will be back on Tuesday for his office hrs.

No class Monday due to holiday

Today's topic - Induction, Section 4.1

Guaranteed it will be on final exam

What sort of problems need to be done by induction?

Ex1 Let n be an arbitrary positive integer, i.e. $n \in \{1, 2, 3, \ldots\}$ (start at 1)

Prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Ex2 Prove that

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

i.e. the sum of the first n odd #'s is n^2.

Ex3 Prove $n^3 - n$ is always divisible by 3.

What is the pattern? You are trying to prove a property about all positive integers.

Here's the technique. See p 265

Suppose $P(n)$ is a statement about all positive integers,
i.e. $P(n)$ might say
n satisfies $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

To prove it true for all positive integers,
Step 1 (Base case)
Prove $P(1)$ is true
Step 2 Inductive Step
Prove the statement
$$P(k) \Rightarrow P(k+1)$$

For Step 2, note this is an "If ..., then ..."
To prove it, one must assume $P(k)$ (this is the inductive hypothesis). From this assumption, prove $P(k+1)$.

denote IH

Why does this work?

Suppose you know $P(1)$ is true and you know that $\forall k$, $P(k) \Rightarrow P(k+1)$ is true.
Let $k=1$. So $P(1) \Rightarrow P(2)$. You already know $P(1)$, so now you have $P(2)$. Then let $k=2$
So you know $P(2)$ and you know
$P(2) \Rightarrow P(3)$. So you get $P(3)$, etc.

Back to examples

Ex1. $P(n)$ is the statement

$$1+2+3+ \cdots + n = \frac{n(n+1)}{2}$$

$P(1)$ is $1 = \frac{1(1+1)}{2}$ or $1=1$.

(Base case is almost always easy. But you **must** do it!)

Now I assume $P(k)$ is true, i.e.

$$1+2+3+ \cdots + k = \frac{k(k+1)}{2}$$

Need to prove $P(k+1)$, i.e.

$$1+2+3+ \cdots + k+(k+1) = \frac{(k+1)(k+2)}{2}.$$

LHS is $\underbrace{[1+2+3+4+ \cdots + k]}+(k+1)$

By IH, this equals

$$\frac{k(k+1)}{2}+(k+1) = \frac{k^2+k}{2} + k+1$$

$$= \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2} = \text{RHS} \checkmark$$

Ex2 Now $P(n)$ is

$$1+3+5+\cdots+(2n-1)=n^2$$

Base case $n=1$

$$1=1^2 \quad ✓$$

I.H. Assume

$$1+3+5+\cdots+(2k-1)=k^2$$

Need to prove

$$\underbrace{1+3+5+\cdots+(2k-1)}+(2(k+1)-1)=(k+1)^2$$

$$LHS = k^2 + 2(k+1)-1$$

$$= k^2+2k+2-1 = k^2+2k+1=(k+1)^2 = RHS ✓$$

Ex3 Now $P(n)$ is

"n^3-n is divisible by 3"

Base case $P(1)$ is

"0 is divisible by 3" ✓

IH Suppose k^3-k is divisible by 3

I must prove $(k+1)^3-(k+1)$ is divisible by 3

$$(k+1)^3=(k^3+3k^2+3k+1)$$

So $(k+1)^3-(k+1)=k^3+3k^2+2k$

$$= k^3-k+3k^2+3k = k^3-k+3(k^2+k)$$

$k^3 - k$ is divisible by 3, by IH.

$3(k^2+k)$ is divisible by 3. So their sum is divisible by 3. $\square$

Ex 4 Prove, for all positive integers that
"$8^n - 2^n$ is divisible by 6." $= P(n)$.

Base Case.

$P(1) = 8^1 - 2^1 = 6$, which is divisible by 6.

Assume $8^k - 2^k$ is divisible by 6. (IH)

We must show

$8^{k+1} - 2^{k+1}$ is divisible by 6.

This one has a trick. Since you have seen the trick, it is fair for tests. *

TRICK $8^{k+1} - 2^{k+1} = 8(8^k - 2^k) + 8\cdot 2^k - 2^{k+1}$

Tell them this

$= 8(8^k - 2^k) + 8\cdot 2^k - 2\cdot 2^k$

$= 8(8^k - 2^k) + 6\cdot 2^k$

Rest is easy.