Lecture 17 – 348

Announcement

I will be away all next week.
The exam will proceed as scheduled.
It will be proctored by Prof. P. Hofstra.
Thursday's lecture will be given by Prof. P. Scott.
It will cover Section 4.1
To make up missing office hours, I will hold a review session before final.

Section 5.4 Binomial Thm (Not on Midterm 2)

Consider the polynomials $(x+y)^n$.
If we expand them out, what are coefficients?

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

Put these #'s in Pascal's triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Pascal's identity

To obtain an entry in the triangle, just add the two #'s above it, ~~and~~ just to the left and right.
Why does this work? Subject of 5.4.

Notation: Instead of $C(n,r)$, we now write $\binom{n}{r}$ and say "n choose r". The word "choose" comes from the fact that $\binom{n}{r}$ is the # of ways of choosing r elements from an n element set.

Thm (Binomial Thm)

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n}y^n$$

Notice also the pattern of the exponents.

Proof: What is the coefficient in front of $x^{n-j}y^{j}$. This amounts to choosing $n-j$ terms in the product

$$\underbrace{(x+y)(x+y)\cdots\cdots(x+y)}_{n \text{ times}}$$

and taking the x from that term. We would then take y's from the remaining j terms. This can be done in $\binom{n}{n-j}$ ways and $\binom{n}{n-j}=\binom{n}{j}$ (last class) $\square$

Ex: What is coefficient of $x^{37}y^{29}$ in $(x+y)^{66}$?

A: $\binom{66}{29}$

Ex: What is coefficient of $x^{14}y^{28}$ in $(3x+7y)^{42}$?

A: Substituting $3x$ for x
$7y$ for y

A: $\binom{42}{28}3^{14}7^{28}$

Thm: $\displaystyle\sum_{k=0}^{n}\binom{n}{k}=2^{n}$

Proof: Take the binomial expansion of $(x+y)^n$
and let $x=y=1$.

Horrible exercise: Try to prove this directly, using only defn of $\binom{n}{k}$.

Thm: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

Proof: let $x=-1$ and $y=1$.

Sim horrible exercise.

Pascal's identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

This is what allows you to generate the triangle.

Proof: Suppose I have a set X w/ $n+1$ elements. Pick an element $a \in X$. There are 2 types of k element subsets of X, those which contain a and those which don't. No subset appears in both lists.

How many k element subsets contain a?

$\binom{n}{k-1}$, since you are choosing k-1 elements from those remaining.

How many k element subsets do not contain a?

$\binom{n}{k}$ by same argument.

Now apply sum rule. $\square$

Another Identity

Vandermonde's Identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

See if you can find a similar proof.