Lecture 15 - 1348

Monday - review for midterm which will be a week from Monday

Last Time 5.2 Pigeonhole principle

p347 If k is a positive integer and n objects, with $n > k$, are placed into k-boxes, there is at least one box with more than one object.

This is obviously equivalent to:

If $f: X \to Y$ is a function with $|X| = n$ and $|Y| = k$, with $n > k$, then there is at least one $y \in Y$ which is the image of 2 distinct elements of X.

Ex1 There are at least 2 people in this class who had the same grade on the midterm

Proof: Let X be the set of students in this class

Let Y = set of all possible grades.

$|X| = 190$

$|Y| = 51$, since $Y = \{0, \frac{1}{2}, 1, 1.5, \ldots, 25\}$

$f: X \to Y$ assigns grades.
$|X| > |Y|$, so there is some $y \in Y$ which is in the image of at least 2 elements of X. □

Ex 2 Suppose I have any 12 positive integers. At least 2 of these #'s have the same remainder when divided by 11.

Why? There are 11 possible remainders:

$0, 1, 2, \ldots, 10$

Ex 3 Show that for every integer n, there is a multiple of n that has only 0's and 1's in its ~~[illegible]~~ decimal expansion.

Solution: Let n be a positive integer. Consider the $n+1$ numbers

$$1, 11, 111, 1111, \ldots, \underbrace{1111\ldots1}_{n+1 \text{ times}}$$

When I divide each of these by n, there are n possible different remainders. Hence, by PHP, 2 must have same remainder. Say that n_1 and n_2 are the #'s and the remainder is R. Then $n_1 - n_2$ has remainder 0.

④ $n_i - n_j$ will consist of just 0's and 1's

Ex: Let $n = 4$

1, 11, 111, 1111, 11111

remainders 1, 3, 3, 3, 3

Pick any 2 with same remainder

$11111 - 111 = 11000$, and is divisible by 4

<u>Generalized PHP</u> If N objects are placed into k boxes, there is at least one box with $\lceil N/k \rceil$ objects.

<u>Note</u> $\lceil N/k \rceil$ means the smallest integer greater than or equal to N/k

Ex $\lceil 1.2 \rceil = 2$ $\lceil 3 \rceil = 3$, etc

<u>Ex</u>: There are 190 students in this class

$\left\lceil \frac{190}{12} \right\rceil = 16$

So there is at least one month out of the 12 with at least 16 students born in that month.

p354 #31 Suppose there are 38 different time periods during which classes can be scheduled. What is the minimum # of rooms necessary?

Solution: Let N be the # of classes and k = # of time slots. Imagine a function that assigns classes to time slots. Then $\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{677}{38} \right\rceil = ?$

It is possible to assign slots in such a way that each slot has this many classes in it, or less. So this is minimum # needed

5.3 Permutations and combinations

Suppose I have four ~~...~~ cards labelled A, B, C, D. I wish to arrange them in a row. How many ways of doing this are there

$\underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

Answer $4 \cdot 3 \cdot 2 \cdot 1 = 4!$

Any arrangement of the 4 cards in a row is called a permutation of the set $\{A, B, C, D\}$

In general, if X is any finite set, a permutation of X, is an arrangement of the elements of X in order.

Thm: If $|X| = n$, there are $n!$ permutations

Now, given the same set, how many ways ~~to~~ are there of picking 2 of the 4 elements, and placing them in order

$$\boxed{4}\,\boxed{3}$$

Answer $4 \cdot 3 = \dfrac{4!}{2!}$

In general, if X is a set with n elements and $r \leq n$ is a positive integer, then an <u>r-permutation of X</u> is a choice of r elements and an arrangement of those r elements into an order. The number of r-permutations is denoted $P(n,r)$

By usual argument

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$
$$= \frac{n!}{(n-r)!} \quad \text{(Why?}$$

Ex: Consider a standard deck of cards (52 cards)

How many 5 card hands can be dealt, when order matters?

$$A: P(52,5) = \frac{52!}{47!}$$

How many ways can the deck be shuffled

$$52! \approx 8.07 \times 10^{67}$$