April 2011 Final 208 – Solutions

1. (A) Revenue is max at its vertex (since it is a parabola) x-coordinate of the vertex is in the middle between x-intercepts: $R(x)=0$ if

$x=0$ or $60x = 2{,}000 \Rightarrow x = \frac{100}{3}$

So the max of $R(x)$ is at $x = \frac{50}{3}$

$p\left(\frac{50}{3}\right) = 2{,}000 - 60 \cdot \frac{50}{3} = \$1{,}000$ per computer

(B) $R(x) = C(x)$

$2{,}000x - 60x^2 = 4{,}000 + 500x$

$\Rightarrow \quad 60x^2 - 1{,}500x + 4{,}000 = 0 \quad / \div 20$

$3x^2 - 75x + 200 = 0$

$$x = \frac{75 \pm \sqrt{(75)^2 - (4)(3)\cdot 200}}{6} = \begin{cases} 3.035 \\ 21.965 \end{cases}$$

Break-even points:

$(3.035,\ 5{,}517.576)$ & $(21.965,\ 14{,}982.424)$

in thousands

or $(3035, \$5517576)$ &
$(21965, \$14982424)$

(C) Loss will occur for the values of $x \in [1, 3.035) \cup (21.965, 25]$ where 1 unit = 1,000 computers or on $[1{,}000;\ 3035) \cup (21965, 25000]$

Profit will occur on the interval:

$(3.035, 21.965)$ or	$(3035, 21965)$
if 1 unit $\overset{?}{=}$ 1,000 computers	1 unit = 1 computer

2. (A) $e^{12} = (e^4)^x e^{x^2} \Leftrightarrow e^{12} = e^{4x+x^2}$

$\Rightarrow 12 = 4x + x^2$ or $x^2 + 4x - 12 = 0$

factors: 6 & -2 $\qquad (x+6)(x-2) = 0$

Hence $\underline{x=-6}$ or $\underline{x=2}$

2(B) $27 = 3^3$ so $(3^3)^{2x} = 3^{x^2-7}$ ③

$\Rightarrow 6x = x^2 - 7$ or $x^2 - 6x - 7 = 0$

factors: -7 & 1 $(x-7)(x+1) = 0$ ✓

$\Rightarrow \underline{x = 7}$ or $\underline{x = -1}$

(C) $\log_b x = \log_b (4)^{3/2} - \log_b (8)^{2/3} + \log_b (2)^2$

$4^{3/2} = (\sqrt{4})^3 = 8$; $8^{2/3} = (\sqrt[3]{8})^2 = 4$

So $x = \dfrac{8 \cdot 4}{4} = \underline{\underline{8}}$

(D) $\log_a x(x-4) = \log_a 21$

$x^2 - 4x = 21 \Rightarrow x^2 - 4x - 21 = 0$

$(x-7)(x+3) = 0$

$\Rightarrow \underline{\underline{x = 7}}$ or $\cancel{x = -3}$

not in the domain of log.

2 (E) $\log_8(x+6) + \log_8(x+4) = 1$

$\log_8(x+6)(x+4) = 1$

$1 = \log_8 8^1$

$(x+6)(x+4) = 8$

$x^2 + 10x + 24 - 8 = 0 \Rightarrow x^2 + 10x + 16 = 0$

$(x+8)(x+2) = 0$

$\Rightarrow x = -8$ or $x = -2$

Check the domain:

$x + 6 > 0$ & $x + 4 > 0$

$x > -6$ & $x > -4$ so

the only solution is $\underline{\underline{x = -2}}$

3. (A) Arithmetic sequence

$S_{60} = 60 \dfrac{f(1) + f(60)}{2} = 30[15 + (-162)] =$

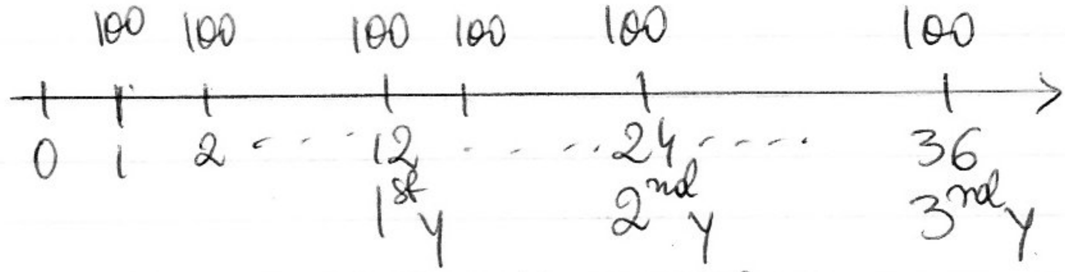
$= \underline{-4,410}$

(B) Geometric: $S_{31} = g(0) \dfrac{r^{31} - 1}{r - 1} = \dfrac{1}{256} \dfrac{4^{31} - 1}{3}$

$r = 4$ $g(0) = 4^{-4} = \dfrac{1}{256}$

$= \underline{\underline{6.0048 \cdot 10^{15}}}$

(4)



Interest earned in the 1st y:

$$I_1 = \underset{1^{st}\,y}{FV} - \underset{\text{deposits}}{12 \cdot 100} = \$\underline{33.56}$$

$$FV_{1st} = 100\,\frac{(1.005)^{12}-1}{0.005} = 1{,}233.56$$

Interest earned in the 2nd year:

$$I_2 = FV_2 - FV_1 - 12 \cdot 100 \stackrel{or}{=}$$

$$= FV_2 - 24 \cdot 100 - I_1 = \$\underline{109.64}$$

$$FV_2 = 100\,\frac{(1.005)^{24}-1}{0.005} = 2{,}543.20$$

Interest earned in the 3rd year:

$$I_3 = FV_3 - FV_2 - 12 \cdot 100 \stackrel{or}{=}$$

$$= FV_3 - 36 \cdot 100 - I_1 - I_2 = \$\underline{190.41}$$

$$FV_3 = 100\,\frac{(1.005)^{36}-1}{0.005} = 3{,}933.61$$

5. Loan $= PV = 129,000$

$i = \frac{0.072}{12} = 0.006$ $\quad n = 20 \cdot 12 = 240$

(A) $129,000 = PMT \, \frac{1-(1.006)^{-240}}{0.006}$

$\Rightarrow PMT = \underline{\$1,015.68}$

Total interest paid: $240 \cdot PMT - PV = \underline{\underline{114,763.2}}$

(B) $PMT^{new} = 1015.68 + 102.41 = 1,118.09$

$129,000 = 1,118.09 \, \frac{1-(1.006)^{-n}}{0.006}$

$0.692251965 = 1-(1.006)^{-n}$

$\Rightarrow (1.006)^{-n} = 1 - 0.6922519\ldots$

$(-n) \ln 1.006 = \ln[\quad]$

$\Rightarrow n = -\frac{\ln 0.307748035}{\ln 1.006} = 197$ months

i.e. 16 years & 5 months

(C) $240 \cdot PMT - 197 \cdot PMT^{new} =$

$= 243,763.20 - 220,263.73 = \underline{\underline{23,499.47}}$

6. Augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 8 & -1 & -18 \\ 2 & 1 & 5 & 8 \\ 2 & 4 & 2 & -4 \end{array}\right] \xrightarrow[R_3^n = R_3 - R_2]{R_1^n = R_1 - R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 7 & -6 & -26 \\ 2 & 1 & 5 & 8 \\ 0 & 3 & -3 & -12 \end{array}\right] \xrightarrow[R_3^n = \frac{1}{3} R_3]{R_2^n = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 7 & -6 & -26 \\ 0 & -13 & 17 & 60 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$-2R_1 \left[\begin{array}{ccc|c} -2 & -14 & 12 & 52 \end{array}\right]$$

$$-7R_3 \left[\begin{array}{ccc|c} 0 & -7 & +7 & +28 \end{array}\right]$$

$$13R_3 \left[\begin{array}{ccc|c} 0 & 13 & -13 & -52 \end{array}\right]$$

$$\xrightarrow[R_2^n = R_2 + 13R_3]{R_1^n = R_1 - 7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & 4 & 8 \\ 0 & 1 & -1 & -4 \end{array}\right] \xrightarrow{R_2^n = \frac{1}{4} R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & -1 & -4 \end{array}\right] \xrightarrow[R_3^n = R_3 + R_2]{R_1^n = R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

So $x_1 = 0$, $x_2 = -2$ & $x_3 = 2$ is the solution of this system

7. (A)

$$M = \begin{array}{c} \\ A \\ M \\ E \end{array}\begin{array}{c} \begin{array}{ccc} A & M & E \end{array} \\ \begin{bmatrix} 0.2 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.1 \end{bmatrix} \end{array}$$

(B) $D = \begin{bmatrix} 10 \\ 20 \\ 15 \end{bmatrix}$ $\qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\begin{array}{c} A \\ M \\ E \end{array}$ in billions \$

$$X = MX + D \quad \text{or} \quad (I - M)X = D$$

(C) $$[I - M \mid D] = \left[\begin{array}{ccc|c} 0.8 & -0.4 & -0.3 & 10 \\ -0.2 & 0.9 & -0.1 & 20 \\ -0.2 & -0.1 & 0.9 & 15 \end{array}\right] \xrightarrow{10 \cdot R_i}$$

$$\left[\begin{array}{ccc|c} 8 & -4 & -3 & 100 \\ -2 & 9 & -1 & 200 \\ -2 & -1 & 9 & 150 \end{array}\right] \xrightarrow[R_2^n = R_2 - R_3]{R_1^n = R_1 + 4R_3} \left[\begin{array}{ccc|c} 0 & -8 & 33 & 700 \\ 0 & 10 & -10 & 50 \\ -2 & -1 & 9 & 150 \end{array}\right]$$

$4R_3 [-8 \;\; -4 \;\; 36 \mid 600]$

$$\xrightarrow[R_3 \leftrightarrow R_1]{R_2^n = \frac{1}{10} R_2} \left[\begin{array}{ccc|c} -2 & -1 & 9 & 150 \\ 0 & 1 & -1 & 5 \\ 0 & -8 & 33 & 700 \end{array}\right] \xrightarrow[R_3^n = R_1 + 8R_2]{R_1^n = R_1 + R_2}$$

$$\rightarrow \left[\begin{array}{ccc|c} -2 & 0 & 8 & 155 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 25 & 740 \end{array}\right] \xrightarrow[R_1^n = -R_1]{R_3^n = \frac{1}{25} R_3} \left[\begin{array}{ccc|c} 2 & 0 & -8 & -155 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & 29.6 \end{array}\right]$$

7. (C) $\xrightarrow[R_2^n = R_1 + R_3]{R_1^n = R_1 + 8R_3} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 81.8 \\ 0 & 1 & 0 & 34.6 \\ 0 & 0 & 1 & 29.6 \end{array}\right]$

$\xrightarrow{R_1^n = \frac{1}{2} R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 40.9 \\ 0 & 1 & 0 & 34.6 \\ 0 & 0 & 1 & 29.6 \end{array}\right]$

They need to produce \$40.9 billion from Agriculture & \$34.6 billion from Manufacturing & \$29.6 — " — from Energy to satisfy the final demands.

8. $\begin{cases} 2x + 2y \geq 4 & L_1 \\ 6x + 4y \leq 36 & L_2 \\ 2x + y \leq 10 & L_3 \\ x, y \geq 0 \end{cases}$

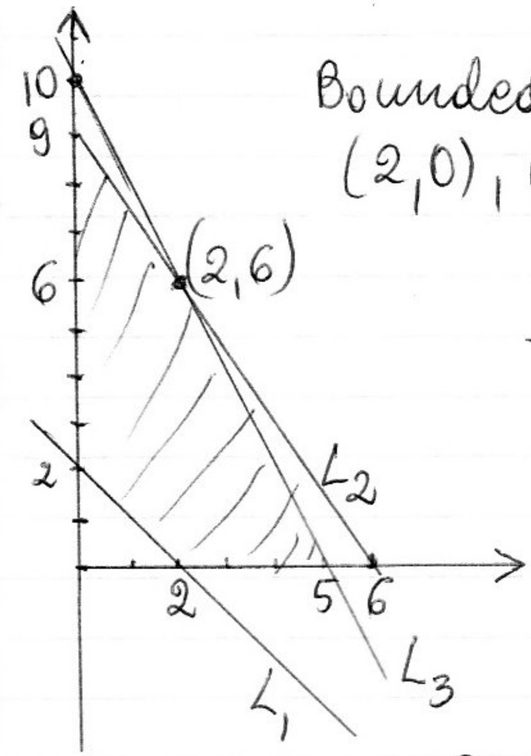
L_1: $x + y = 2$
$(2,0)$, $(0,2)$

L_2: $3x + 2y = 18$
$x = 0 \Rightarrow y = 9$
$y = 0 \Rightarrow x = 6$

L_3: $2x + y = 10$
$(0,10)$ & $(5,0)$

$L_2 \cap L_3$: $\begin{cases} 3x + 2y = 18 & / \cdot (-1) \\ 2x + y = 10 & / \cdot 2 \end{cases}$

$-3x - 2y = -18$
$\underline{4x + 2y = 20}$ sum: $x = 2$ & $y = 6$



Bounded region with corner points:
$(2,0)$, $(5,0)$, $(2,6)$, $(0,9)$ & $(0,2)$

Corner Point	$P = 60x + 20y$
$(2,0)$	120
$(5,0)$	$300 \leftarrow$ max
$(2,6)$	$120 + 120 = 240$
$(0,9)$	180
$(0,2)$	$40 \leftarrow$ min.

The max. of $P = 300$ at $(5,0)$ & the min. is 40 at $(0,2)$

9. There are 2-Gold, x-Silver & 2x-Bronze

$$2 + x + 2x = 20 \Rightarrow 3x = 18 \Rightarrow x = 6$$

$G \Leftrightarrow 2 \qquad S \Leftrightarrow 6 \qquad B \Leftrightarrow 12$ courses

(A) $G + S = 8$ courses so 8 selections are possible $\quad C_{8,1} = \binom{8}{1} = \underline{\underline{8}}$

(B) $C_{12,1} \cdot C_{6,1} \cdot C_{2,1} = 12 \cdot 6 \cdot 2 = \underline{\underline{144}}$

10. $A = \{$at least two choose the same book$\}$

$A' = \{$all 5 choose different book$\}$ $\quad n(A') = P_{10,5}$

$n(S) = 10 \cdot 10 \cdot \ldots \cdot 10 = 10^5$

$$Pr(A) = 1 - Pr(A') = 1 - \frac{10!}{5!\,10^5}$$