April 2012 Final - Solutions

1. (A) Supply 9,800 → price 1.94

9,400 → price 1.82

$(9{,}800;\ 1.94) = (x_1, y_1)$ or (x_1, p_1)

$(9{,}400;\ 1.82) = (x_2, y_2) = (x_2, p_2)$

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{p_2 - p_1}{x_2 - x_1} = \frac{1{,}82 - 1{,}94}{9{,}400 - 9{,}800} = \frac{0{,}12}{400}$$

So $m_1 = 0.0003$

$p^S = 0.0003\,x + b_1$

$1{,}82 = 0.0003(9{,}400) + b_1 \Rightarrow b_1 = -2{,}82 + 1{,}82 = -1$

So $\boxed{p = 0{,}0003\,x - 1}$ is a price - supply equation

(B) Demand: 9,300 → price 1,94

9,500 → price 1,82

$(x_1, p_1) = (9{,}300;\ 1.94)$ & $(x_2, p_2) = (9{,}500;\ 1{,}82)$

$$m_2 = \frac{1{,}82 - 1{,}94}{9{,}500 - 9{,}300} = \frac{-0{,}12}{200} = -0.0006$$

$p^D = -0{,}0006\,x + b_2$

$1,82 = -0,0006(9,500) + b_2$

$\Rightarrow b_2 = 1,82 + 5,7 = 7,52$ &

$\boxed{p = -0,0006x + 7,52}$ is a price-demand equation

(c) $p^S = p^D$

$0,0003x - 1 = -0,0006x + 7,52$

$0,0009x = 8,52 \Rightarrow x = 9,466,667$ million bushels

& the price is \$1,84

So at a price of 1,84 the demand equals the supply $= 9,466.667$ million bushels

2. (A) $3^{3x-x^2} = \frac{1}{81}$

$81 = 3^4$ so $\frac{1}{81} = 3^{-4}$

Hence $3^{3x-x^2} = 3^{-4} \Rightarrow 3x - x^2 = -4.$

$3x - x^2 + 4 = 0$ or $x^2 - 3x - 4 = 0$

factors: $-4, 1$ $(x-4)(x+1) = 0$

$\underline{\underline{x = 4}}$ or $\underline{\underline{x = -1}}$

(B) $(81)^{2x} = (9)^{x^2-12}$

$(9^2)^{2x} = 9^{x^2-12} \Rightarrow 2(2x) = x^2 - 12$

$0 = x^2 - 12 - 4x$

$x^2 - 4x - 12 = 0$ factors: $-6, 2$

$(x-6)(x+2) = 0$

$\underline{\underline{x = 6}}$ or $\underline{\underline{x = -2}}$

2 (C)

$$3\log_b 2 + \frac{1}{2}\log_b 25 - \log_b 20 = \log_b x$$

$$\log_b 2^3 + \log_b \sqrt{25} - \log_b 20 = \log_b x$$

$$\log_b \frac{8 \cdot 5}{20} = \log_b x \Rightarrow \underline{\underline{x = 2}}$$

(D) $\log_5(x+6) + \log_5(x+2) = 1$

$$\log_5(x+6)(x+2) = \log_5 5$$

$\Rightarrow (x+6)(x+2) = 5 \qquad x^2 + 8x + 12 - 5 = 0$

$x^2 + 8x + 7 = 0$ factors: 7 & 1

$(x+7)(x+1) = 0 \Rightarrow x = -7$ or $x = -1$

But we have $x+6 > 0$ & $x+2 > 0$

from the domain of logarithm

so $x = -7$ is <u>not</u> a good solution

The only solution is $\underline{\underline{x = -1}}$

since $-1+6 > 0$ & $-1+2 > 0$

2 (E) $\log_4(x^2+x+4)=2$

$\log_4(x^2+x+4)=\log_4 4^2$

$\Rightarrow \quad x^2+x+4=4^2 \qquad x^2+x+4-16=0$

$x^2+x-12=0$

Again factors: 4 & -3 give x

so $(x+4)(x-3)=0 \Rightarrow \underline{\underline{x=-4}}$ or $\underline{\underline{x=3}}$

Both are good: check:

$(-4)^2+(-4)+4 = 16 > 0$ in the domain!

3. (A) Arithmetic sequence

$f(0)=16 \qquad f(1)=-22+16=-8$

so $d=-22 \qquad f(40)=-22(40)+16=-864$

$S_{41}=41\cdot\left(\frac{f(0)+f(40)}{2}\right)=41\left(\frac{16-864}{2}\right)$

$=41\cdot(-424)=-\underline{\underline{17,384}}$

3 (B) Geometric sequence

$g(0) = 5$, $g(1) = 5 \cdot (0.8)$ ...

$r = 0.8$

$S_{30} = 5\left[1 + 0.8 + (0.8)^2 + \dots + (0.8)^{29}\right] =$

$= 5 \dfrac{1-(0.8)^{30}}{1-0.8} = 5 \dfrac{1-(0.8)^{30}}{0.2} =$

$= 25\left[1-(0.8)^{30}\right] = 25(0.99876206) = \underline{\underline{24.969}}$

4. You are saving for this trip — you do __not__ take a loan !

$4 \cdot 24 = 96$ deposits to accumulate

$i = \dfrac{0.072}{24} = 0.003$ per $\frac{1}{2}$ month

(A)

$FV = 100,000$ $\qquad PMT = X$

$100,000 = X \dfrac{(1.003)^{96} - 1}{0.003}$

$X = \dfrac{100,000 \cdot (0.003)}{(1.003)^{96} - 1} = 900.4077 \simeq \underline{\underline{900.41}}$

4(B) Total Interest $= 100{,}000 - 96 \cdot PMT =$

$= 100{,}000 - 96 \cdot 900.41 = 13{,}560.64$

(C) $PMT = 800 \quad n = ? \quad , FV = 100{,}000$

$$100{,}000 = 800 \left(\frac{(1.003)^n - 1}{0.003} \right)$$

$$\frac{100{,}000 \cdot (0.003)}{800} = (1.003)^n - 1$$

$$\frac{3}{8} + 1 = (1.003)^n \Rightarrow 1.375 = (1.003)^n$$

Take ln of both sides:

$$\ln 1.375 = n \ln 1.003$$

$$\Rightarrow n = \frac{\ln 1.375}{\ln 1.003} = 106.31 \text{ half-months}$$

so $\underline{\underline{107}}$ half-months

It takes 4 years & 11 payments more so 4 years & 5.5 months

5. $PV = 275{,}000 - 55{,}000 = 220{,}000$ a loan

$n = 30 \cdot 12 = 360$ months (payments)

$i = \frac{0.06}{12} = 0{,}005$ per month

(A) $X = PMT$

$$220{,}000 = X \, \frac{1-(1.005)^{-360}}{0{,}005}$$

$$\Rightarrow X = \frac{(0{,}005) \cdot 220{,}000}{1-(1.005)^{-360}} = \underline{\underline{\$1{,}319.01}}$$

(B) Total Interest Paid $= 360 \cdot PMT - 220{,}000 =$

$= 360 \cdot 1{,}319.01 - 220{,}000 =$

$= \underline{254{,}843.60}$

(C) Interest paid in the 1st payment =

$= 0.005 \cdot$ Loan $= (0{,}005) \cdot 220{,}000 =$

$= \underline{\underline{1{,}100}}$

The rest of the payment goes for reducing the balance of the loan so PMT − interest = $\underline{\underline{219.01}}$ only

6. x = # of airplanes with 10 passangers

A y = # of —"— —"— 15 —"—

z = # of —"— —"— 20 —"—

$$\begin{cases} x+y+z=12 \\ 10x+15y+20z=220 \quad / \div 5 \end{cases}$$

$$\begin{cases} x+y+z=12 \quad / \cdot (-2) \\ 2x+3y+4z=44 \end{cases}$$

$$\begin{cases} -2x-2y-2z=-24 \\ 2x+3y+4z=44 \end{cases}$$ sum them

$$y+2z=20 \Rightarrow y=20-2z$$

$$x+(20-2z)+z=12$$

$$x+20-z=12$$

$$\underline{x=z-8}$$

$x \geq 0$ & $y \geq 0$ & $z \geq 0$

$z-8 \geq 0$ $\qquad$ $20-2z \geq 0$

$z \geq 8$ $\qquad$ $20 \geq 2z$

$10 \geq z$

6. So z could be only : 8, 9 or 10

A

x	0	1	2
y	4	2	0
z	8	9	10

B To minimize the monthly cost we need to calculate it for each solutions:

I: $14{,}000(4) + 16{,}000(8) = 184{,}000$

II $8{,}000 + 14{,}000(2) + 16{,}000\cdot(9) = 180{,}000$

III $8{,}000(2) + 16{,}000(10) = \underline{176{,}000}$

the best option !

7.

$$\begin{array}{cc} & \begin{array}{ccc} T & A & F \end{array} \\ \text{input} \begin{array}{c} T \\ A \\ F \end{array} & \begin{bmatrix} 0.3 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.1 \end{bmatrix} \\ & \text{output} \end{array}$$

7.(B) $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad D = \begin{bmatrix} 20 \\ 5 \\ 10 \end{bmatrix}$

$$X = MX + D \quad \text{or} \quad (I - M)X = D.$$

(c) $I - M = \begin{bmatrix} 0.7 & -0.1 & -0.3 \\ -0.2 & 0.9 & -0.2 \\ -0.1 & -0.1 & 0.9 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0.7 & -0.1 & -0.3 & 20 \\ -0.2 & 0.9 & -0.2 & 5 \\ -0.1 & -0.1 & 0.9 & 10 \end{array}\right] \xrightarrow[R_3^n = -10R_3]{R_i^n = 10R_i,\ i=1,2} \left[\begin{array}{ccc|c} 7 & -1 & -3 & 200 \\ -2 & 9 & -2 & 50 \\ 1 & 1 & -9 & -100 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & -9 & -100 \\ -2 & 9 & -2 & 50 \\ 7 & -1 & -3 & 200 \end{array}\right] \xrightarrow[R_3^n = R_3 - 7R_1]{R_2^n = R_2 + 2R_1}$$

$2R_1 \left[\begin{array}{ccc|c} 2 & 2 & -18 & -200 \end{array}\right]$

$-7R_1 \left[\begin{array}{ccc|c} -7 & -7 & 63 & 700 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 1 & -9 & -100 \\ 0 & 11 & -20 & -150 \\ 0 & -8 & 60 & 900 \end{array}\right] \xrightarrow{R_3^n = \frac{1}{8}R_3} \left[\begin{array}{ccc|c} 1 & 1 & -9 & -100 \\ 0 & 11 & -20 & -150 \\ 0 & -1 & 7.5 & 112.5 \end{array}\right]$$

Rewritten with changed order!

$$\left[\begin{array}{ccc|c} 1 & 1 & -9 & -100 \\ 0 & -1 & 7{,}5 & 112{,}5 \\ 0 & 11 & -20 & -150 \end{array}\right] \xrightarrow[R_3^n = R_3 + 11R_2]{R_1^n = R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1{.}5 & 12{.}5 \\ 0 & -1 & 7{.}5 & 112{.}5 \\ 0 & 0 & 62{,}5 & 1{,}087{,}5 \end{array}\right]$$

$11R_2 [0 \; -11 \; 82{,}5 \,|\, 1{,}237{,}5]$

divide by 62.5
& $R_2^n = -R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1.5 & 12.5 \\ 0 & 1 & -7{,}5 & -112.5 \\ 0 & 0 & 1 & 17.4 \end{array}\right] \xrightarrow[R_2^n = R_2 + 7{,}5R_3]{R_1^n = R_1 + 1.5R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 38.6 \\ 0 & 1 & 0 & 18 \\ 0 & 0 & 1 & 17.4 \end{array}\right]$$

$1{,}5R_3 [0 \; 0 \; 1.5 \,|\, 26.1]$

$7.5R_3 [0 \; 0 \; 7.5 \,|\, 130{,}5]$

They need \$38,5 million from T,
\$18 — " — from A &
\$17,4 — " — from F.

8. $$\begin{cases} 2x + y \leq 40 & L_1 \\ 20x + 2y \geq 36 & L_2 \\ 6x + 15y \geq 108 & L_3 \\ x \geq 0,\; y \geq 0 \end{cases}$$

L_1: $2x+y=40$

$x=0 \Rightarrow y=40$

$y=0 \Rightarrow x=20$

L_2: $20x+2y=36 \;/ \div 2$

$10x+y=18$

$x=0 \Rightarrow y=18$

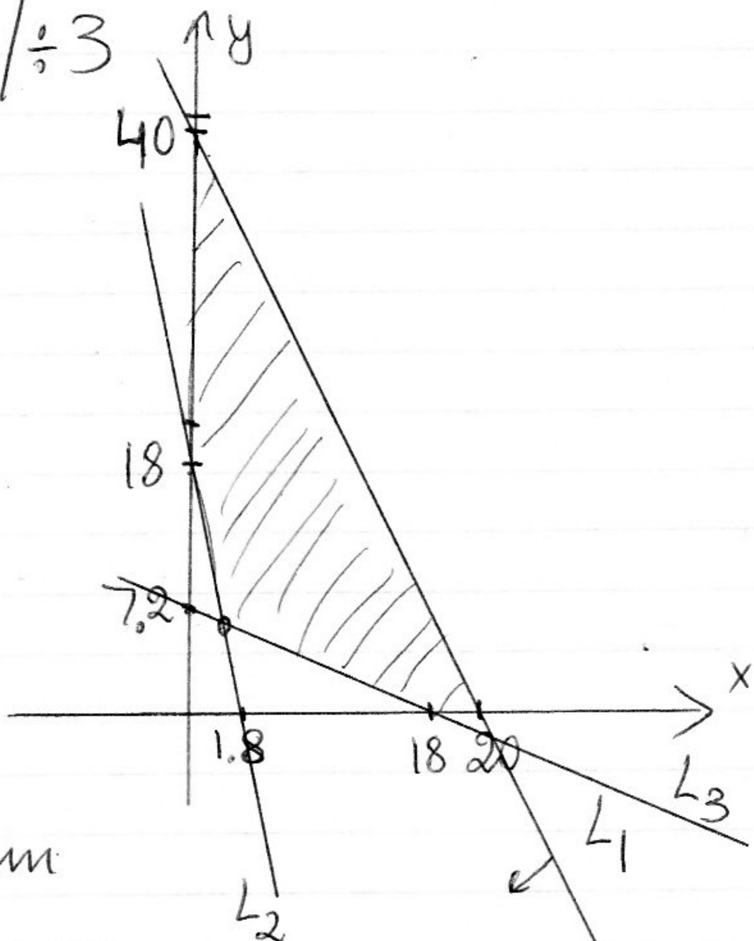
$y=0 \Rightarrow x=1.8$

L_3: $6x+15y=108 \;/ \div 3$

$2x+5y=36$

$x=0 \Rightarrow y=7.2$

$y=0 \Rightarrow x=18$



We need $L_2 \cap L_3$:

$$\begin{cases} 10x+y=18 & /\cdot 5 \\ 2x+5y=36 & /\cdot(-1) \end{cases}$$

$$\begin{array}{r} +50x+5y=90 \\ -2x-5y=-36 \\ \hline 48x=54 \Rightarrow x=1.125 \end{array}$$ sum

$10(1.125)+y=18 \Rightarrow y=6.75$

So we have bounded feasible region with the corner points: $(0,40)$, $(0,18)$, $(1.125, 6.75)$, $(18,0)$ & $(20,0)$

Now we check the value of P in those points:

Corner point	$P(x,y) = 15x + 5y$
$(0, 40)$	$5 \cdot 40 = 200$
$(0, 18)$	$5 \cdot 18 = 90$
$(1.125, 6.75)$	$15 \cdot (1.125) + 5 \cdot (6.75) = 50{,}625$ ← min
$(18, 0)$	$15 \cdot (18) = 270$
$(20, 0)$	$15 \cdot 20 = 300$ ← max

P attains max. of 300 at $(20, 0)$
& min. of 50.625 at $(1.125, 6.75)$

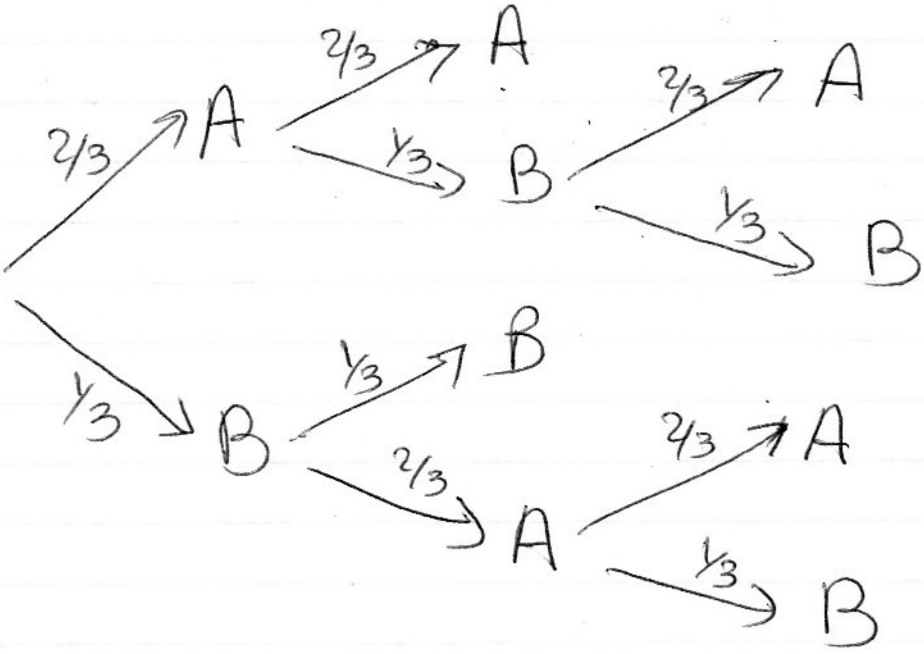
9. (A) In total there are
$5 + 8 + 12 = 25$ stores

$$C_{25,10} = \binom{25}{10} = \frac{25!}{10!\,15!}$$

(B) $$C_{5,2} \cdot C_{8,4} \cdot C_{12,4} = \binom{5}{2}\binom{8}{4}\binom{12}{4} =$$

$$= \frac{5!}{2!\,3!} \cdot \frac{8!}{4!\,4!} \cdot \frac{12!}{4!\,8!}$$

10. A means Ann wins a set
B — " — Barbara — " —



Ist set   IInd set   IIIrd set

(A) Ann wins in three matches possible:

$$Pr\{A \text{ wins}\} = \frac{2}{3}\cdot\frac{2}{3} \underset{\text{or}}{+} \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} =$$

$$= \frac{12+4+4}{27} = \underline{\underline{\frac{20}{27}}}$$

(B) $$Pr\{3 \text{ sets are played}\} = \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} +$$

$$+ \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{4}{27} + \frac{2}{27} + \frac{4}{27} + \frac{2}{27} =$$

$$= \frac{12}{27} = \underline{\underline{\frac{4}{9}}}$$

(c) $\Pr\{A \text{ wins the } 1^{st} \ \& \ A \text{ wins the match}\} =$

$$= \tfrac{2}{3} \cdot \tfrac{2}{3} + \tfrac{2}{3} \cdot \tfrac{1}{3} \cdot \tfrac{2}{3} = \frac{4}{9} + \frac{4}{27} =$$

$$= \frac{12+4}{27} = \tfrac{16}{27}$$

$\Pr\{B \text{ wins the } 1^{st} \ \& \ B \text{ wins the match}\} =$

$$= \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \tfrac{2}{3} \cdot \frac{1}{3} = \frac{3}{27} + \frac{2}{27} = \tfrac{5}{27}$$

$\Pr\{\text{player who wins the } 1^{st} \text{ wins the match}\} =$

$$= \tfrac{16}{27} + \tfrac{5}{27} = \frac{21}{27} = \underline{\underline{\tfrac{7}{9}}}$$