

Questions 1-5 are multiple choice.

1. [3] The general solution of $xy'' + y' + 7xy = 0$ for $x > 0$ is

- (a) $c_1J_0(\sqrt{7}x) + c_2J_0(\sqrt{7}x)\ln(x)$ (b) $c_1J_0(\sqrt{7}x) + c_2Y_0(\sqrt{7}x)$
 (c) $c_1J_{\sqrt{7}}(x) + c_2J_{-\sqrt{7}}(x)$ (d) $c_1J_{\sqrt{7}}(x) + c_2Y_{\sqrt{7}}(x)$ (e) None of these

2. [3] The general solution of $x^2y'' + xy' + (5x^2 - 9)y = 0$ for $x > 0$ is

- (a) $c_1J_3(\sqrt{5}x) + c_2J_{-3}(\sqrt{5}x)$ (b) $c_1J_3(\sqrt{5}x) + c_2Y_3(\sqrt{5}x)$
 (c) $c_1J_{\sqrt{5}}(3x) + c_2J_{-\sqrt{5}}(3x)$ (d) $c_1J_{\sqrt{5}}(3x) + c_2Y_{\sqrt{5}}(3x)$ (e) None of these

3. [2] At $x = 20$, the Fourier Cosine series of the 8-periodic function $f(x) = \begin{cases} 3, & 0 < x < 2 \\ 0, & 2 < x < 4 \end{cases}$ converges to

- (a) 0 (b) $\frac{3}{2}$ (c) $-\frac{3}{2}$ (d) 3 (e) None of these

4. [2] The function $x^3 \sin(x) + x^2 \cos(x)$ is

- (a) even; (b) odd; (c) neither .

5. [2] If f is a periodic function of period 3, and $f(x) = 2x^2$, $3 > x \geq 0$, then $f(97)$ is

- (a) 18; (b) 8; (c) -8; (d) 2; (e) 0.

Answers: b, b, a, a, d.

6. [18 marks] Find the full Fourier series of the 2π -periodic function $f(x) = \begin{cases} x, & 0 < x \leq \pi \\ 1, & \pi < x \leq 2\pi \end{cases}$

Give the first three terms of the series explicitly, i.e, the terms $\frac{a_0}{2}$, $a_1 \cos\left(\frac{n\pi x}{L}\right)$ and $b_1 \sin\left(\frac{n\pi x}{L}\right)$ with the appropriate a_0, a_1, b_1, n and L .

Solution:

$$a_0 = \frac{1}{\pi} \left\{ \int_0^\pi x \, dx + \int_\pi^{2\pi} 1 \, dx \right\} = \frac{\pi}{2} + 1 = \frac{\pi + 2}{2}.$$

For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi} \left\{ \int_0^\pi x \cos(nx) \, dx + \int_\pi^{2\pi} \cos(nx) \, dx \right\} = \frac{1}{\pi} \left\{ \frac{x}{n} \sin(nx) + \frac{1}{n^2} \cos(nx) \right\}_0^\pi + \frac{1}{\pi} \left\{ \frac{1}{n} \sin(nx) \right\}_\pi^{2\pi} = \\ &= \frac{1}{\pi} \left[\frac{1}{n^2} \cos(n\pi) - \frac{1}{n^2} \right] + 0 = \frac{1}{n^2\pi} [(-1)^n - 1]. \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \left\{ \int_0^\pi x \sin(nx) \, dx + \int_\pi^{2\pi} \sin(nx) \, dx \right\} = \frac{1}{\pi} \left\{ -\frac{x}{n} \cos(nx) + \frac{1}{n^2} \sin(nx) \right\}_0^\pi + \frac{1}{\pi} \left\{ -\frac{1}{n} \cos(nx) \right\}_\pi^{2\pi} = \\ &= \frac{1}{\pi} \left[-\frac{\pi}{n} \cos(n\pi) \right] - \frac{1}{n\pi} \left[\cos(2n\pi) - \cos(n\pi) \right] = \frac{1}{n\pi} \left[-\pi(-1)^n - 1 + (-1)^n \right]. \end{aligned}$$

To give the first three terms, we will need $\frac{a_0}{2} = \frac{\pi + 2}{4}$, $a_1 = -\frac{2}{\pi}$, $b_1 = \frac{\pi - 2}{\pi}$.

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The full Fourier series is given by

$$\frac{\pi + 2}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi} \left((-1)^n - 1 \right) \cos(nx) + \frac{1}{n\pi} \left(-\pi(-1)^n - 1 + (-1)^n \right) \sin(nx) =$$

$$\frac{\pi + 2}{4} - \frac{2}{\pi} \cos(x) + \frac{\pi - 2}{\pi} \sin(x) + \dots$$