(A)

MAT 1322 A W2011 Wednesday, Feb. 2nd 8:30–9:50 Prof. Desjardins

MIDTERM TEST 1

Max = 20

Solutions

Student Number: ______________________________

•Time: 80 min.

• Only basic scientific calculators are permitted (non-graphing, non-programmable, no integration or differentiation capabilities). Notes or books are not permitted.

• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

• The problems require complete and clearly presented solutions and carry part marks if there is substantial correct work towards the solution.

• There are five questions worth four marks each.

A

1. (a) Consider the integral $\int_1^4 \frac{3}{(x-2)^{5/3}}\,dx$. Does it converge or diverge? If it converges, give its value.

$$\int_1^4 \frac{3}{(x-2)^{5/3}}\,dx = \int_1^2 \frac{3}{(x-2)^{5/3}}\,dx + \int_2^4 \frac{3}{(x-2)^{5/3}}\,dx$$

then $\displaystyle\int_1^2 \frac{3}{(x-2)^{5/3}}\,dx = \lim_{t\to 2^-} \int_1^t \frac{3}{(x-2)^{5/3}}\,dx$

$$= \lim_{t\to 2^-} -\frac{9}{2}(x-2)^{-2/3}\Big|_1^t$$

$$= \lim_{t\to 2^-} -\frac{9}{2}\left(\frac{1}{(t-2)^{2/3}} - \frac{1}{(-1)^{2/3}}\right) = -\infty \text{ (diverges)}$$

$\therefore \displaystyle\int_1^4 \frac{3}{(x-2)^{5/3}}\,dx$ $\boxed{\text{diverges}}$

(b) Use the Comparison Test to determine if the integral $\int_2^\infty \frac{3+\sin x}{x^3+5x}\,dx$ converges or diverges.

for all x $\quad -1 \le \sin x \le 1$, so $\quad 2 \le 3+\sin x \le 4$

then for all $x \geq 2$, $\quad \frac{3+\sin x}{x^3+5x} \le \frac{4}{x^3+5x} < \frac{4}{x^3}$

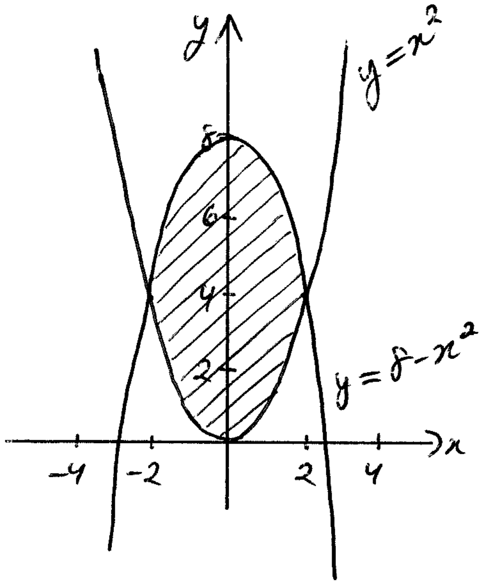
(since $x^3+5x > x^3$, we'll have $\frac{1}{x^3+5x} < \frac{1}{x^3}$)

so $\displaystyle\int_2^\infty \frac{3+\sin x}{x^3+5x}\,dx < \int_2^\infty \frac{4}{x^3}\,dx$ which is known to converge $(p=3>1)$

$\therefore \displaystyle\int_2^\infty \frac{3+\sin x}{x^3+5x}\,dx$ $\boxed{\text{converges}}$

Ⓐ

2. Sketch the region bounded by the curves $y = x^2$ and $y = 8 - x^2$. What is the area of the region?



points of intersection:

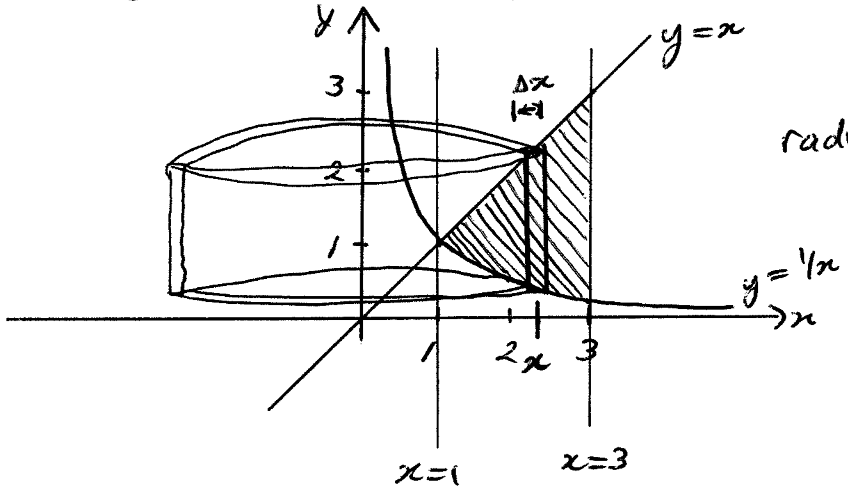
$$x^2 = 8 - x^2$$
$$2x^2 = 8$$
$$x = \pm 2$$

so the area is

$$A = \int_{-2}^{2} \left((8-x^2) - x^2\right)dx$$
$$= \int_{-2}^{2} (8 - 2x^2)\,dx$$
$$= 2\int_{0}^{2} (8 - 2x^2)\,dx \qquad \text{(by symmetry)}$$
$$= 2\left(8x - \tfrac{2}{3}x^3 \Big|_0^2\right)$$
$$= 2\left(16 - 16/3\right)$$
$$= 2\left(32/3\right)$$
$$= \boxed{64/3} \simeq \boxed{21.33}$$

Ⓐ

3. Use the method of cylindrical shells to find the volume of the solid obtained when the region bounded by $y = x$, $y = 1/x$, $x = 1$ and $x = 3$ is rotated around the y-axis. Include a sketch of the region and a typical cylinder (with dimensions) in your solution.



radius of shell is x

height is $x - 1/x$

thickness is Δx

so volume of shell is $2\pi x\,(x - 1/x)\,\Delta x$

and the volume of the solid is

$$V = \int_1^3 2\pi x\,(x - 1/x)\,dx$$

$$= 2\pi \int_1^3 (x^2 - 1)\,dx$$

$$= 2\pi \left(\tfrac{1}{3}x^3 - x \Big|_1^3 \right)$$

$$= 2\pi \left[(9-3) - (1/3 - 1) \right]$$

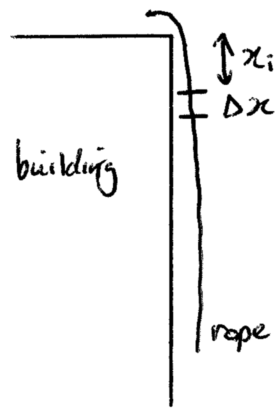
$$= 2\pi\,(7 - 1/3)$$

$$= 2\pi\,(20/3)$$

$$= \boxed{40\pi/3} \simeq \boxed{41.89}$$

A

4. A heavy rope of length 12 m has a density of 1.5 kg/m and is hanging over the edge of a tall building. How much work is done pulling the rope to the top of the building? The acceleration of gravity is $g = 9.8 \text{ m/s}^2$. Define clearly all the variables that enter into your solution and provide a drawing which shows their meaning.



Chop the rope into lengths Δx m.
Consider the length x_i m below the top of the building

this length of rope has mass $1.5\Delta x$ kg and weight $1.5\, g\, \Delta x = 14.7 \Delta x$ N

this length must be lifted x_i m, so the work done on this piece is $W_i = 14.7\, x_i \Delta x$ J

so the total work done is $W \approx \sum_i W_i = \sum_i 14.7 x_i \Delta x$ J

take the limit as $\Delta x \to 0$ to get

$$W = \lim_{\Delta x \to 0} \sum_i 14.7 x_i \Delta x = \int_0^{12} 14.7\, x\, dx$$

$$= 14.7 \int_0^{12} x\, dx$$

$$= 14.7 \left(\tfrac{1}{2} x^2 \Big|_0^{12} \right)$$

$$= 14.7\, (144/2)$$

$$= \boxed{1058.4 \text{ J}}$$

(A)

5. (a) Give the integral for the average value of the function $f(x) = \arcsin x$ on the interval $[0, 1/2]$ and then evaluate it.

$$f_{ave} = \frac{1}{1/2 - 0}\int_0^{1/2} \arcsin x \, dx = 2\int_0^{1/2} \arcsin x \, dx$$

$$\left(\begin{aligned} u &= \arcsin x \\ du &= \frac{1}{\sqrt{1-x^2}}\,dx \\ dv &= dx \\ v &= x \end{aligned} \right)$$

$$= 2\left(x \arcsin x \Big|_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx \right)$$

$$= 2\left(x \arcsin x + \sqrt{1-x^2} \Big|_0^{1/2} \right)$$

$$= 2\left[\left(\tfrac{1}{2} \arcsin(1/2) + \sqrt{3/4} \right) - (0+1) \right]$$

$$= 2\left((1/2)(\pi/6) + \sqrt{3}/2 - 1 \right)$$

$$= \boxed{\pi/6 + \sqrt{3} - 2} \simeq \boxed{0.2556}$$

(b) Solve the initial value problem: $(1+x^2)\dfrac{dy}{dx} = 2y\,,\;\; y(0) = 3\,.$

separate the variables $\quad \dfrac{dy}{y} = \dfrac{2}{1+x^2}\,dx$

integrate on both sides $\quad \displaystyle\int \frac{dy}{y} = \int \frac{2}{1+x^2}\,dx + C$

we get $\quad \ln|y| = 2\arctan x + C$

exponentiate $\quad y = e^{2\arctan x + C} = Ke^{2\arctan x}$

then $y(0) = 3 \Rightarrow 3 = Ke^0 \Rightarrow K = 3$

$\therefore \boxed{y(x) = 3e^{2\arctan x}}$

(B)

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• The problems require complete and clearly presented solutions and carry part marks if there is substantial correct work towards the solution.

• There are five questions worth four marks each.

(B)

1. (a) Consider the integral $\int_1^5 \frac{3}{(x-4)^{7/3}}\,dx$. Does it converge or diverge? If it converges, give its value.

$$\int_1^5 \frac{3}{(x-4)^{7/3}}\,dx = \int_1^4 \frac{3}{(x-4)^{7/3}}\,dx + \int_4^5 \frac{3}{(x-4)^{7/3}}\,dx$$

where $\int_1^4 \frac{3}{(x-4)^{7/3}}\,dx = \lim_{t\to4^-} \int_1^t \frac{3}{(x-4)^{7/3}}\,dx$

$$= \lim_{t\to4^-} -\frac{9}{4}(x-4)^{-4/3}\Big|_1^t$$

$$= \lim_{t\to4^-} -\frac{9}{4}\left(\frac{1}{(t-4)^{4/3}} - \frac{1}{(-3)^{4/3}}\right) = -\infty \text{ (diverges)}$$

$\therefore \int_1^5 \frac{3}{(x-4)^{7/3}}\,dx$ $\boxed{\text{diverges}}$

(b) Use the Comparison Test to determine if the integral $\int_1^\infty \frac{4-\cos x}{x^4+3x}\,dx$ converges or diverges.

for all x $\quad -1 \le \cos x \le 1 \Rightarrow 3 \le 4-\cos x \le 5$

for all $x \geq 1$ $\quad x^4+3x > x^4 \Rightarrow \frac{1}{x^4+3x} < \frac{1}{x^4}$

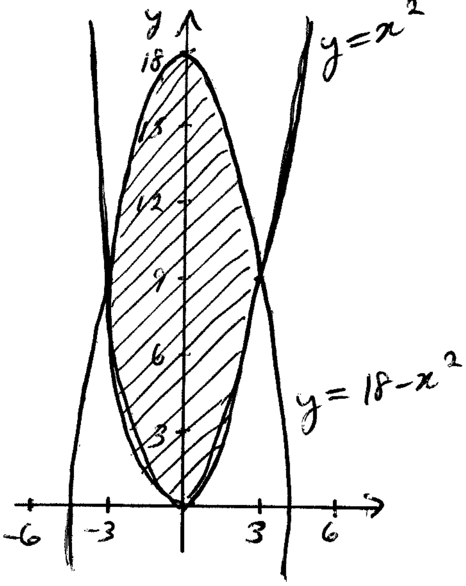
and so $\quad \frac{4-\cos x}{x^4+3x} \le \frac{5}{x^4+3x} < \frac{5}{x^4}$

thus $\quad \int_1^\infty \frac{4-\cos x}{x^4+3x}\,dx < \int_1^\infty \frac{5}{x^4}\,dx$ converges $(p=4>1)$

$\therefore \int_1^\infty \frac{4-\cos x}{x^4+3x}\,dx$ $\boxed{\text{converges}}$

B

2. Sketch the region bounded by the curves $y = x^2$ and $y = 18 - x^2$. What is the area of the region?



$$18 - x^2 = x^2$$
$$2x^2 = 18$$
$$x = \pm 3$$

$$A = \int_{-3}^{3} ((18-x^2) - x^2)\,dx$$

$$= 2\int_0^3 (18 - 2x^2)\,dx$$

$$= 2\left(18x - \tfrac{2}{3}x^3 \Big|_0^3\right)$$

$$= 2(54 - 18)$$

$$= \boxed{72}$$

(B)

3. Use the method of cylindrical shells to find the volume of the solid obtained when the region bounded by $y = x$, $y = 1/x$, $x = 1$ and $x = 2$ is rotated around the y-axis. Include a sketch of the region and a typical cylinder (with dimensions) in your solution.

diagram as in Version A, ending at $x = 2$

dimensions of shell the same, volume same

so $V = \int_1^2 2\pi x \,(x - 1/x)\,dx$

$= 2\pi \int_1^2 (x^2 - 1)\,dx$

$= 2\pi \left(\frac{1}{3}x^3 - x \Big|_1^2 \right)$

$= 2\pi \left[(8/3 - 2) - (1/3 - 1) \right]$

$= 2\pi \, (7/3 - 1)$

$= 2\pi \, (4/3)$

$= \boxed{8\pi/3} \simeq \boxed{8.3776}$

B

4. A heavy rope of length 10 m has a density of 1.25 kg/m and is hanging over the edge of a tall building. How much work is done pulling the rope to the top of the building? The acceleration of gravity is $g = 9.8$ m/s^2. Define clearly all the variables that enter into your solution and provide a drawing which shows their meaning.

diagram as in version A

mass of length is $1.25\,\Delta x$ kg

weight of length is $1.25 g\,\Delta x = 12.25\,\Delta x$ N

work is $W \approx \sum_i W_i = \sum_i 12.25\, x\, \Delta x$ J

so $W = \int_0^{10} 12.25\, x\, dx$

$= 12.25 \left(\frac{1}{2} x^2 \Big|_0^{10}\right)$

$= \boxed{612.5 \text{ J}}$

(B)

5. (a) Give the integral for the average value of the function $f(x) = \arctan x$ on the interval $[0, \sqrt{3}]$ and then evaluate it.

$$f_{ave} = \frac{1}{\sqrt{3}-0} \int_0^{\sqrt{3}} \arctan x \, dx = \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}} \arctan x \, dx$$

$$\begin{pmatrix} u = \arctan x \\ du = \frac{1}{1+x^2} dx \\ dv = dx \\ v = x \end{pmatrix}$$

$$= \frac{1}{\sqrt{3}} \left(x \arctan x \Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2} dx \right)$$

$$= \frac{1}{\sqrt{3}} \left(x \arctan x - \frac{1}{2} \ln(1+x^2) \Big|_0^{\sqrt{3}} \right)$$

$$= \frac{1}{\sqrt{3}} \left[\left(\sqrt{3} \arctan(\sqrt{3}) - \frac{1}{2} \ln(4)\right) - \left(0 - \frac{1}{2} \ln(1)\right) \right]$$

$$= \frac{1}{\sqrt{3}} \left(\sqrt{3} \left(\pi/3\right) - \ln 2 \right)$$

$$= \boxed{\pi/3 - \frac{1}{\sqrt{3}} \ln 2} \simeq \boxed{0.6470}$$

(b) Solve the initial value problem: $\sqrt{1-x^2}\,\frac{dy}{dx} = 3y$, $y(0) = 4$.

$$\frac{dy}{y} = \frac{3}{\sqrt{1-x^2}} dx$$

$$\int \frac{dy}{y} = \int \frac{3}{\sqrt{1-x^2}} dx + C$$

$$\ln|y| = 3 \arcsin x + C$$

$$y = K e^{3 \arcsin x}$$

$y(0) = 4$ $\qquad 4 = Ke^0 \Rightarrow K = 4$

$$\therefore \boxed{y(x) = 4e^{3 \arcsin x}}$$