MATH 1402B **Term test #2** **13 February 2012**

Maximum score: 30 marks

Non-programmable calculators are permitted. This test is closed book.
Supply your answers on this sheet, but TA's have extra paper if you need it.

PLEASE PRINT	____________ First name	____________ Last name	____________ Student number

Please show your work where appropriate!

1. [4] Let $z = f(x, y) = x^3 + y^2 - 2xy + 7x - 8y + 2$. **Determine the (*x*, *y*) coordinates of the critical point(s)**. *You do not have to determine* whether these are a max/min/saddle.

$\frac{\partial z}{\partial x} = 0 \Rightarrow 3x^2 - 2y + 7 = 0$; $\frac{\partial z}{\partial y} = 0 \Rightarrow 2y - 2x - 8 = 0$

$\Rightarrow y = x + 4$

$\therefore 3x^2 - 2(x+4) + 7 = 0$

$3x^2 - 2x - 1 = 0 \Rightarrow (3x+1)(x-1) = 0$

$\therefore x = 1 \Rightarrow y = 5$

$x = -1/3 \Rightarrow y = 3\,2/3 = 11/3$

$\therefore$ (1,5) and $(-1/3, 11/3)$ are critical points.

2. [6] Let $z = f(x, y) = x^3 + y^3 - 27x - 12y + 25$. — mistake in original question, (which doesn't matter).
This function has 4 critical points whose (*x*, *y*) coordinates are: (1, 2), (–1, –2), (1, –2) and (–1, 2).
Determine the nature of each of these 4 points (max/min/saddle).

$\frac{\partial z}{\partial x} = 3x^2 - 27 \Rightarrow \frac{\partial^2 z}{\partial x^2} = 6x$

$\frac{\partial z}{\partial y} = 3y^2 - 12 \Rightarrow \frac{\partial^2 z}{\partial y^2} = 6y$

Also note that: $\frac{\partial^2 z}{\partial x \partial y} = 0 \Rightarrow D = \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right)$

$\left.\frac{\partial^2 z}{\partial x^2}\right|_{(1,2)} = 6 > 0$ and $\left.\frac{\partial^2 z}{\partial y^2}\right|_{(1,2)} = 12 > 0$ Min. ✓ $(D > 0)$ confirmed!

$\left.\frac{\partial^2 z}{\partial x^2}\right|_{(-1,-2)} = -6 < 0$ and $\left.\frac{\partial^2 z}{\partial y^2}\right|_{(-1,-2)} = -12 < 0$ Max. ✓ $(D > 0)$

$\left.\frac{\partial^2 z}{\partial x^2}\right|_{(-1,2)} = -6 < 0$ and $\left.\frac{\partial^2 z}{\partial y^2}\right|_{(-1,2)} = 12 > 0$ saddle ✓ $D < 0$

$\left.\frac{\partial^2 z}{\partial x^2}\right|_{(1,-2)} = 6 > 0$ and $\left.\frac{\partial^2 z}{\partial y^2}\right|_{(1,-2)} = -12 < 0$ saddle ✓ $D < 0$

3. [5] Let $z = f(x, y) = y^2 + 4x^2 - 2xy^2$ and $y = g(x) = -2x^2 + 3$. **Determine** $\boldsymbol{dz/dx}$ by any method.

Method 1: Direct substitution:

$$z = f(x,y) = f(x, g(x)) = (-2x^2+3)^2 + 4x^2 - 2x(-2x^2+3)^2$$

$$\therefore \frac{dz}{dx} = 2(-2x^2+3)(-4x) + 8x - 2x(2(-2x^2+3)(-4x)) - 2(-2x^2+3)^2$$

$$\ldots = (-2x^2+3)(-8x + 16x^2 - 2(-2x^2+3)) + 8x$$

$$\therefore \frac{dz}{dx} = y\ (16x^2 - 8x - 2y) + 8x$$

Method 2: Formula for the total derivative:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} \ ; \quad \frac{\partial z}{\partial x} = 8x - 2y^2$$

$$\frac{\partial z}{\partial y} = 2y - 4xy$$

$$\frac{dy}{dx} = g'(x) = -4x$$

$$\therefore \frac{dz}{dx} = 8x - 2y^2 + (2y - 4xy)(-4x)$$

$$= 8x - 2y^2 - 8xy + 16x^2y$$

$$= y(16x^2 - 8x - 2y) + 8x$$

4. [5] Let $z = f(x, y) = x^2 - 2xy + y^2 + 12x - 4y + 3$. Use the differential dz to estimate the change in z (i.e. Δz) when moving from $(x, y) = (1, 2)$ to $(1.05, 1.9)$. Compare the result with the actual value of Δz.

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\frac{\partial z}{\partial x} = 2x - 2y + 12 \ ; \quad \frac{\partial z}{\partial y} = -2x + 2y - 4$$

Also $dx = 0.05$ and $dy = -0.1$

$$\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(1,2)} = 10 \ ; \quad \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(1,2)} = -2$$

$$\therefore dz = 10 \cdot 0.05 + (-2)(-0.1) = 0.7$$

Now, compare dz with:

$\Delta z =$

$f(1.05, 1.9) - f(1,2)$

$f(1.05, 1.9) = 8.7225$

$f(1,2) = 8$

$\boxed{\therefore \Delta z = 0.7225}$

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5. [5] Let $3xy-(x^2-y^2)^2=x+y^2$. Determine dy/dx by any method.

Method 1: Let $3xy-(x^2-y^2)^2-x-y^2=0=f(x,y)$

Then : $\frac{dy}{dx}=-\frac{f_x}{f_y}$, with : $f_x=3y-2(x^2-y^2)(2x)-1$

$\ldots=3y-1-4x(x^2-y^2)$

and : $f_y=3x+4y(x^2-y^2)-2y$

$\therefore \frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{3y-1-4x(x^2-y^2)}{3x-2y+4y(x^2-y^2)}$

Method 2: differentiate both sides w.r.t. x, then solve for $y'=\frac{dy}{dx}$:

$$\frac{d}{dx}\left(3xy-(x^2-y^2)^2\right)=\frac{d}{dx}\left(x+y^2\right)$$

$$3y+3xy'-2(x^2-y^2)\cdot(2x-2yy')=1+2yy'$$

$$3xy'-4x(x^2-y^2)+4y(x^2-y^2)y'=1-3y+2yy'$$

$$y'\left(3x+4y(x^2-y^2)-2y\right)=1-3y+4x(x^2-y^2)$$

$$y'=\frac{1-3y+4x(x^2-y^2)}{3x-2y+4y(x^2-y^2)}$$, which

is algebraically equivalent to the first answer.

6. [5] Let $z=f(x,y)=y^2+2x^2-2xy^2$ and $y=g(x)=x-5$. Determine the (x, y) coordinates of the constrained optima (there are 2).

Method 1 : Direct substitution :

$z=f(x,g(x))=(x-5)^2+2x^2-2x(x-5)^2=(x-5)^2+2x^2-2x^3+20x^2-50x=0$

$\frac{dz}{dx}=0\Rightarrow\quad \underbrace{2(x-5)}_{2x-10}+4x-6x^2+40x-50=0$

$-6x^2+46x-60=0$

$\therefore x=\frac{-46\pm\sqrt{46^2-4(-6)(-60)}}{2(-6)}$

$x=\frac{-46\pm\sqrt{676}}{-12}=\frac{-46\pm 26}{-12}$

$=6$ OR $5/3$

then: $y=x-5=1$ OR $-10/3$

$\therefore (x,y)=(6,1)$ OR $(5/3,-10/3)$

Method 2: set $\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{dy}{dx} = g'(x) = 1$

$f(x,y) = y^2 + 2x^2 - 2xy^2 \Rightarrow f_x = 4x - 2y^2$

$f_y = 2y - 4xy$

$\therefore -\frac{f_x}{f_y} = \frac{4x-2y^2}{4xy-2y} = 1 \Rightarrow 4x - 2y^2 = 4xy - 2y$ — ①

Work with ① and $y = x-5$ to determine the (x,y) coordinates of the critical points:

$4x - 2y^2 = 4xy - 2y$ and $y = x-5$ yields:

$4x - 2(x-5)^2 = 4x(x-5) - 2(x-5)$

$4x - 2x^2 + 20x - 50 = 4x^2 - 20x - 2x + 10$

$-6x^2 - 46x - 60 = 0 \Rightarrow$ this is the same quadratic eq. as before.

Method 3: Lagrange multipliers

set $\mathcal{L} = f + \lambda g$, where $f(x,y) = y^2 + 2x^2 - 2xy^2$ and $g(x,y) = y - x + 5 = 0$

$\therefore \mathcal{L} = f(x,y) + \lambda g(x,y)$

$\mathcal{L} = y^2 + 2x^2 - 2xy^2 + \lambda(y - x + 5)$

Now set up 3 equations:

$\frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow 4x - 2y^2 - \lambda = 0$ — ①

$\frac{\partial \mathcal{L}}{\partial y} = 0 \Rightarrow 2y - 4xy + \lambda = 0$ — ②

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow y - x + 5 = 0$ — ③ this is $g(x,y) = 0$ (always)

from ①: $\lambda = 4x - 2y^2$ — ①'

from ②: $\lambda = 4xy - 2y$ — ②'

To eliminate λ, set: ①' = ②' and then work with ③, and you're back in familiar territory ...