Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Calculus III for Engineers
MAT 2322A - Fall 2016
Midterm I
Professor: Victor G. LeBlanc
Time limit: 80 minutes. Closed books

Name: Solutions ID Number: Version 1

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Instructions

- **The only calculators which are allowed are Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
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- Questions 1 to 3 are multiple choice. These questions are worth 2 points each and no partial marks are possible. **Please write your answers in the corresponding boxes in the grid below entitled "Answers to multiple choice Qs".**
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- Do not unstaple the test. Good luck!

Answers to multiple choice Qs

1	2	3
C	A	F

Grid below is used for grading
(do not write in this grid)

MCQ	4	5	6	Total
/6	/6	/7	/6	/25

1. For all possible unit vectors $\vec{u}$, what is the largest possible value of $D_{\vec{u}}f(3,2)$, where $f(x,y) = x^2 - 3y^2$?

A. $8\sqrt{7}$

B. $7\sqrt{6}$

C. $6\sqrt{5}$

D. $5\sqrt{4}$

E. $4\sqrt{3}$

F. $3\sqrt{2}$

$\max D_{\vec{u}} f(3,2) = \|\vec{\nabla} f(3,2)\|$

$f_x = 2x \quad f_y = -6y$

$\vec{\nabla} f(3,2) = 6\vec{i} - 12\vec{j}$

$\|\vec{\nabla} f(3,2)\| = \sqrt{6^2+12^2}$

$= \sqrt{36+144}$

$= \sqrt{180} = 3\sqrt{20}$

$= 6\sqrt{5}$

2. Which of the following statements is true concerning the critical points of the function $f(x,y) = x^4 - 4x^3 + 4x^2 + y^2$?

A. $(0,0)$ is a local minimum, $(1,0)$ is a saddle, $(2,0)$ is a local minimum

B. f has no critical points

C. $(0,0)$ is a local maximum, $(1,0)$ is a saddle, $(2,0)$ is a local maximum

D. $(0,0)$ is a saddle, $(1,0)$ is a local maximum, $(2,0)$ is a saddle

E. $(0,0)$ is a local maximum, $(1,0)$ is a local minimum, $(2,0)$ is a local maximum

F. None of the above

$f_x = 4x^3 - 12x^2 + 8x = 4x(x^2-3x+2)$

$= 4x(x-1)(x-2)$

$f_y = 2y$

C.P. $(0,0)$, $(1,0)$, $(2,0)$

$f_{xx} = 12x^2 - 24x + 8$, $f_{yy} = 2$

$f_{xy} = 0 = f_{yx}$

$D(0,0) = 16$

$f_{yy}(0,0) = 2 > 0$

local min

$D(1,0) = (-4)(2) = -8 < 0$

$(1,0)$ saddle

$D(2,0) = 8 \cdot 2 = 16 > 0$

$f_{yy}(2,0) > 0$

local min

3. For the composite function $z(u,v) = f(x(u,v), y(u,v))$, which of the following correspond to the chain rule for $\dfrac{\partial z}{\partial v}$?

A. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$

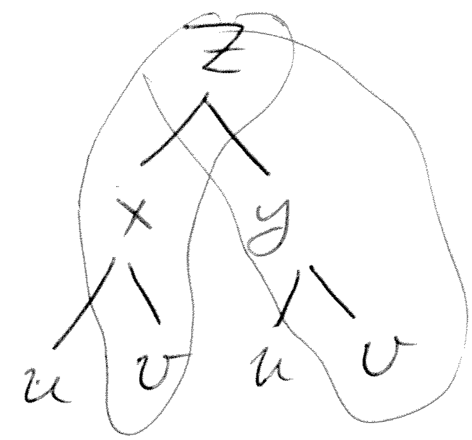
B. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial v} + \dfrac{\partial z}{\partial v}$

C. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

D. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v}$

E. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

F. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$



$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

4. Compute the tangent plane to the graph of the function $z = f(x, y) = x \cos y + y \sin x$ at the point $(x, y, z) = (\pi/2, \pi/2, \pi/2)$.

$f_x = \cos y - y \cos x \qquad f_x(\pi/2, \pi/2) = \cos \pi/2 - \frac{\pi}{2} \cos \pi/2$
$= 0$

$f_y = -x \sin y + \sin x \qquad f_y(\pi/2, \pi/2) = -\frac{\pi}{2} \sin \frac{\pi}{2} + \sin \pi/2$
$= 1 - \pi/2$

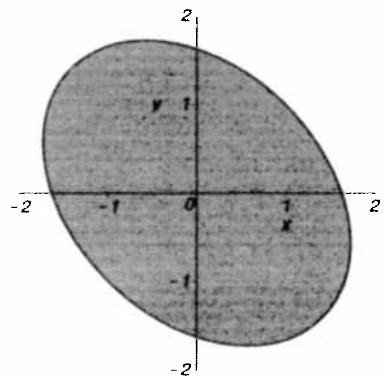
Tangent plane eqn

$$Z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$Z = \pi/2 + 0 + (1 - \pi/2)(y - \pi/2)$$

$$\boxed{Z = \pi/2 + (1 - \pi/2)(y - \pi/2)}$$

5. Find the global extrema of the function $f(x, y) = x^2 + y^2$ over the elliptical domain $D = \{ (x, y) \in \mathbb{R}^2 \mid 3x^2 + 2xy + 3y^2 \leq 8\}$ sketched below. For part of your solution, you must use the method of Lagrange multipliers.



Step 1: Critical pts

$f_x = 2x, \quad f_y = 2y$

Only critical pt is $(x,y) = (0,0)$, which belongs to D. Keep it.

Step 2: Boundary analysis using Lagrange multipliers

$$\boxed{\begin{gathered} \vec{\nabla} f = \lambda \vec{\nabla} g \\ 3x^2 + 2xy + 3y^2 = 8 \end{gathered}}$$

$$g_x = 6x + 2y, \quad g_y = 2x + 6y$$

$$2x = \lambda(6x+2y) \Longrightarrow \lambda = \frac{2x}{6x+2y}$$

$$2y = \lambda(2x+6y)$$

$$3x^2 + 2xy + 3y^2 = 8$$

$$2y = \frac{2x}{6x+2y}(2x+6y)$$

$$2y(6x+2y) = 2x(2x+6y)$$

$$\cancel{12xy} + 4y^2 = 4x^2 + \cancel{12xy}$$

$$y^2 = x^2 \Rightarrow y = \pm x$$

$y = x$

$$3x^2 + 2x \cdot x + 3x^2 = 8$$

$$8x^2 = 8$$

$$x = \pm 1$$

$\boxed{\begin{gathered} (1,1) \\ (-1,-1) \end{gathered}}$ Keep

$y = -x$

$$3x^2 + 2x(-x) + 3(-x)^2 = 8$$

$$4x^2 = 8$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

$\boxed{\begin{gathered} \sqrt{2}, -\sqrt{2} \\ -\sqrt{2}, \sqrt{2} \end{gathered}}$ Keep

Step 3:

(x,y)	$f(x,y)$
$(0,0)$	0
$(1,1)$	2
$(-1,-1)$	2
$(\sqrt{2},-\sqrt{2})$	4
$(-\sqrt{2},\sqrt{2})$	4

global min

global max

6. Compute the double integral of the function $f(x, y) = 12\,x^2y^3$ over the rectangle

$$R = \{\,(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1, 0 \leq y \leq 2\}$$

$$\iint_R f\,dA = \int_0^1 \int_0^2 12x^2y^3\,dy\,dx$$

$$= \int_0^1 \left(\frac{12x^2y^4}{4} \Big/_0^2 \right) dx = \int_0^1 \left(3x^2y^4 \Big/_0^2 \right) dx$$

$$= \int_0^1 \left(3x^2 \cdot 2^4 - 3x^2 \cdot 0^4 \right) dx = \int_0^1 48x^2\,dx$$

$$= \frac{48x^3}{3} \Big/_0^1 = 16x^3 \Big/_0^1 = 16 - 0 = \boxed{16}$$

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Answers to multiple choice Qs

1	2	3
E	D	A

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MCQ	4	5	6	Total
/6	/6	/7	/6	/25

1. For all possible unit vectors $\vec{u}$, what is the largest possible value of $D_{\vec{u}}f(1,2)$, where $f(x,y) = 2x^2 - y^2$?

$\max D_{\vec{u}} f(1,2) = \|\vec{\nabla} f(1,2)\|$

A. $8\sqrt{6}$

$f_x = 4x \quad f_y = -2y$

B. $7\sqrt{5}$

$\vec{\nabla} f(1,2) = 4\vec{i} - 4\vec{j}$

C. $6\sqrt{4}$

D. $5\sqrt{3}$

$\|\vec{\nabla} f(1,2)\| = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$

E. $4\sqrt{2}$

F. $3\sqrt{1}$

2. Which of the following statements is true concerning the critical points of the function $f(x,y) = x^4 - 4x^3 + 4x^2 - y^2$?

A. $(0,0)$ is a local minimum, $(1,0)$ is a saddle, $(2,0)$ is a local minimum

B. f has no critical points

C. $(0,0)$ is a local maximum, $(1,0)$ is a saddle, $(2,0)$ is a local maximum

D. $(0,0)$ is a saddle, $(1,0)$ is a local maximum, $(2,0)$ is a saddle

E. $(0,0)$ is a local maximum, $(1,0)$ is a local minimum, $(2,0)$ is a local maximum

F. None of the above

$f_x = 4x^3 - 12x^2 + 8x = 4x(x^2-3x+2) = 4x(x-1)(x-2)$

$f_y = -2y$

C.P. $(0,0), (1,0), (2,0)$

$f_{xx} = 12x^2 - 24x + 8, \quad f_{yy} = -2 \quad f_{xy} = f_{yx} = 0$

$D(0,0) = 8\cdot(-2) = -16 < 0$

$(0,0)$ saddle

$D(1,0) = (-4)(-2) = 8 > 0$

$f_{yy} < 0$

$(1,0)$ local max

$D(2,0) = 8(-2) = -16 < 0$

$(2,0)$ saddle

3. For the composite function $z(u,v) = f(x(u,v), y(u,v))$, which of the following correspond to the chain rule for $\dfrac{\partial z}{\partial u}$?

A. $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$

B. $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial v} + \dfrac{\partial z}{\partial v}$

C. $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

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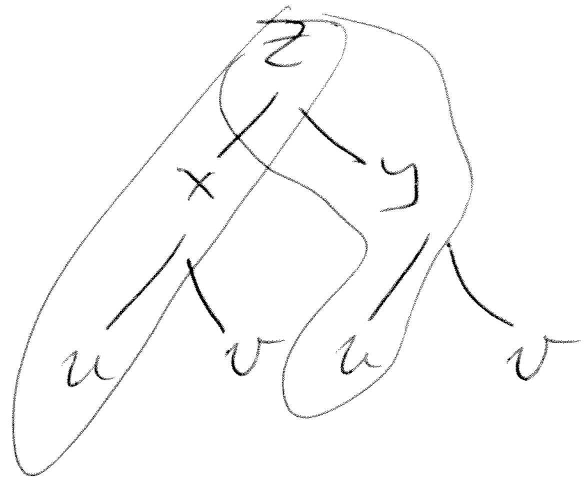

$\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$

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$f_x = -y\sin x + \sin y$ $\qquad f_x(\pi/2, \pi/2) = -\frac{\pi}{2}\sin\frac{\pi}{2} + \sin\frac{\pi}{2}$

$= 1 - \pi/2$

$f_y = \cos x + x\cos y$ $\qquad f_y(\pi/2, \pi/2) = \cos\pi/2 + \pi/2\cos\frac{\pi}{2} = 0$

Tangent plane eq

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$z = \frac{\pi}{2} + (1-\pi/2)(x-\pi/2) + 0$

$$\boxed{z = \frac{\pi}{2} + (1-\pi/2)(x-\pi/2)}$$

5. Find the global extrema of the function $f(x,y) = x^2 + y^2$ over the elliptical domain $D = \{ (x,y) \in \mathbb{R}^2 \mid 3x^2 - 2xy + 3y^2 \leq 8 \}$ sketched below. For part of your solution, you must use the method of Lagrange multipliers.

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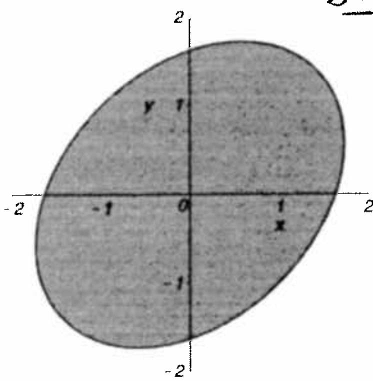

Step 2: Boundary analysis using Lagrange multipliers

$$\boxed{\begin{aligned} \vec{\nabla} f &= \lambda \vec{\nabla} g \\ 3x^2 - 2xy + 3y^2 &= 8 \end{aligned}}$$

$g_x = 6x - 2y \qquad g_y = -2x + 6y$

$2x = \lambda(6x - 2y) \longrightarrow \lambda = \dfrac{2x}{6x-2y}$

$2y = \lambda(-2x + 6y)$

$3x^2 - 2xy + 3y^2 = 8$

$2y = \dfrac{2x}{6x-2y}(-2x+6y)$

$2y(6x-2y) = 2x(-2x+6y)$

$\cancel{12xy} - 4y^2 = -4x^2 + \cancel{12xy}$

$y^2 = x^2 \Rightarrow$

$y = \pm x$

$y = x$	$y = -x$
$3x^2 - 2x^2 + 3x^2 = 8$	$3x^2 + 2x^2 + 3x^2 = 8$
$4x^2 = 8$	$8x^2 = 8$
$x^2 = 2$	$x^2 = 1$
$x = \pm\sqrt{2}$	$x = \pm 1$
$\boxed{(\sqrt{2}, \sqrt{2}) \quad (-\sqrt{2}, -\sqrt{2})}$	$\boxed{(1,-1) \quad (-1,1)}$

Step 3:

(x,y)	$f(x,y)$
$(0,0)$	0 global min
$(\sqrt{2}, \sqrt{2})$	4 global max
$(-\sqrt{2}, -\sqrt{2})$	4
$(1,-1)$	2
$(-1,1)$	2

6. Compute the double integral of the function $f(x,y) = 12\,x^3y^2$ over the rectangle

$$R = \{\,(x,y) \in \mathbb{R}^2 \,|\, 0 \le x \le 2, 0 \le y \le 1\}$$

$$\iint_R f\,dA = \int_0^1 \int_0^2 12x^3y^2\,dx\,dy = \int_0^1 \left(\frac{12x^4}{4} y^2 \Big/_0^2 \right) dy$$

$$= \int_0^1 \left(3x^4y^2 \Big/_0^2 \right) dy = \int_0^1 (3 \cdot 2^4 y^2 - 3 \cdot 0^4 \cdot y^2)\,dy =$$

$$\int_0^1 48\,y^2\,dy = \frac{48y^3}{3} \Big/_0^1 = 16y^3 \Big/_0^1 = 16 - 0$$

$$= \boxed{16}$$