Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Midterm 2 for MAT 1339B (Winter 2018)
Introduction to Calculus and Vectors
March 23, 2018

Duration : 80 minutes

Professor : Rachid Bentoumi

NAME : ______________

STUDENT NUMBER : ______________

You must sign below.

Version 2

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- **This is a closed book examination.**
- **Only Faculty standard calculators are permitted.**
- **There are 5 questions.**
- **The exam is out of 40 points**

1. For the following equations, determine the value of x

(a) (2 points) $9 + 4\ln(x) = 17$

$$4 \ln(x) = 17 - 9 = 8$$

$$\ln(x) = \frac{8}{4} = 2$$

$$e^{\ln(x)} = e^2 \quad \Rightarrow \quad x = e^2$$

(b) (2 points) $3^{x+2} = 4^{2-x}$

$$\ln(3^{x+2}) = \ln(4^{2-x})$$

$$\Rightarrow (x+2)\ln(3) = (2-x)\ln(4)$$

$$\Rightarrow x\ln(3) + 2\ln(3) = 2\ln(4) - x\ln(4)$$

$$\Rightarrow x\ln(3) + x\ln(4) = 2\ln(4) - 2\ln(3)$$

$$\Rightarrow x(\ln(3) + \ln(4)) = 2\ln(4) - 2\ln(3)$$

$$\Rightarrow x = \frac{2\ln(4) - 2\ln(3)}{\ln(4) + \ln(3)}$$

2. (4 points) Determine the equation of the tangent line to the curve of $y = 5^x$ at the point $x = -2$.

$$y = f'(x_0)(x - x_0) + f(x_0) \qquad f(x) = 5^x, \quad x_0 = -2$$

We need to evaluate $f'(-2)$ and $f(2)$

First, $* \; f(-2) = 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

$$* \; f'(x) = (5^x)' = 5^x \ln(5)$$

$$\Rightarrow f'(-2) = 5^{-2}\ln(5) = \frac{\ln(5)}{25}$$

So, $y = f'(-2)(x - (-2)) + f(-2)$

$$= \frac{\ln(5)}{25}(x+5) + \frac{1}{25}$$

3. Find the derivatives of the following functions :

(a) (3 points) $f(x) = 3\sin^2(2x-4) - 2\cos^2(3x+1)$

$f'(x) = 3\left(\sin^2(2x-4)\right)' - 2\left(\cos^2(3x+1)\right)'$

$= 3(2)\sin'(2x-4)\sin^{2-1}(2x-4) - 2(2)\cos'(3x+1)\cos^{2-1}(3x+1)$

$= 6(2x-4)'\cos(2x-4)\sin(2x-4) - 4(-1)(3x+1)'\sin(3x+1)\cos(3x+1)$

$= 6(2)\cos(2x-4)\sin(2x-4) + 4(3)\sin(3x+1)\cos(3x+1)$

$= 12\cos(2x-4)\sin(2x-4) + 12\sin(3x+1)\cos(3x+1)$

(b) (3 points) $h(x) = 3e^{\tan x} + \ln(\cos x) - \sin(1)$

$h'(x) = 3\left(e^{\tan(x)}\right)' + \ln'(\cos(x)) - \sin'(1)$

$= 3\tan'(x)\,e^{\tan(x)} + \dfrac{\cos'(x)}{\cos(x)} - 0$

$= 3\sec^2(x)\,e^{\tan(x)} + \dfrac{-\sin(x)}{\cos(x)}$

$= 3\sec^2(x)\,e^{\tan(x)} - \dfrac{\sin(x)}{\cos(x)}$

(c) (3 points) $y = e^{2x} - e^{-2x}$

$y' = (e^{2x})' - (e^{-2x})'$

$= (2x)'e^{2x} - (-2x)'e^{-2x}$

$= 2e^{2x} - (-2)e^{-2x}$

$= 2e^{2x} + 2e^{-2x}$

4. A bacterial colony's population is modelled by the function

$$P(t) = 60\, e^{0.8t}$$

where P is the number of bacteria after t days.

(a) (1 point) What is the initial population of the bacterial colony's?

$$P_0 = P(0) = 60\, e^{0.8(0)} = 60\, e^0 = 60$$

(b) (1 point) Determine the bacterial population after 3 days?

At $t=3$, we have $P(3) = 60\, e^{0.8(3)} = 661.39 \simeq 662$

(c) (2 points) How long will it take for the population to reach 10 times its initial population?

We need to solve: $P(t) = 10 P_0 \Rightarrow \cancel{P_0}\, e^{0.8t} = 10 \cancel{P_0}$

$$\Rightarrow e^{0.8t} = 10$$

$$\Rightarrow \ln\left(e^{0.8t}\right) = \ln(10)$$

$$\Rightarrow 0.8t = \ln(10)$$

$$\Rightarrow t = \frac{\ln(10)}{0.8} = 2.879 \text{ days}$$

(d) (2 points) Compute the rate of change of this population at the third day?

$$P'(t) = \left(60\, e^{0.8t}\right)' = 60\,(0.8t)'\, e^{0.8t} = 60\cdot(0.8)\, e^{0.8t}$$
$$= 48\, e^{0.8t}$$

At $t = 3$ days, we have

$$P'(3) = 48\, e^{0.8(3)}$$
$$= 529.1125$$

5. Consider the following function $f(x) = \frac{3x+1}{2-x}$

(a) (1 point) Determine the domain of f.

$2 - x \neq 0 \Rightarrow x \neq 2$. So that, $D_f = (-\infty, 2) \cup (2, \infty)$

(b) (2 points) Find, if it is possible, the coordinates of the intersections of f with x-axis and y-axis.

x-Intercept: we solve $f(x) = 0$. So, $3x+1=0$ if $x = \frac{-1}{3}$

y-intercept.: set $x=0$ and compute $f(0) = \frac{3(0)+1}{2-0} = \frac{1}{2}$

$f(x)$ crosses x-axis at $(-\frac{1}{3}, 0)$

$f(x)$ crosses y-axis at $(0, \frac{1}{2})$

(c) (2 points) Determine, if it is possible, the vertical and horizontal asymptotes.

H. A $\lim_{x \to \pm\infty} \frac{3x+1}{2-x} = \frac{3}{-1} = -3$

Hence, $y = -3$ is a H.A

V. A $\lim_{x \to 2^-} \frac{3x+1}{2-x} = \frac{3(2)+1}{0^+} = \frac{7}{0^+} = +\infty$

$\lim_{x \to 2^+} \frac{3x+1}{2-x} = \frac{3(2)+1}{0^-} = \frac{7}{0^-} = -\infty$

Hence, $x = 2$ is a V.A

(d) (3 points) Calculate the first derivative of f and find all possible critical points.

$$f'(x) = \left(\frac{3x+1}{2-x}\right)' = \frac{3(2-x) - (3x+1)(-1)}{(2-x)^2}$$

$$= \frac{6-3x+3x+1}{(2-x)^2} = \frac{7}{(2-x)^2} > 0$$

There is no x such that $f'(x)=0$ since $f'(x)>0$. In addition $f'(2)$ does not exist. However, $2 \notin D_f \Rightarrow$ No C.P for $f(x)$

(e) (3 points) Calculate the second derivative of f and find all possible inflection points.

$$f''(x) = \left(\frac{7}{(2-x)^2}\right)' = 7\left((2-x)^{-2}\right)' = 7(-2)(2-x)'(2-x)^{-3}$$

$$= -14(-1)(2-x)^{-3} = \frac{14}{(2-x)^3}$$

There is no x such that $f''(x)=0$.

There is no inflection point for $f(x)$.

(f) (3 points) Build the variation table.

x	$-\infty$	2	∞
$f'(x)$	$+$	$\nexists$	$+$
f	$\nearrow$	V.A	$\nearrow$
$f''(x)$	$+$	$\nexists$	$-$
Concavity of f	$\cup$	V.A	$\cap$

(g) (3 points) Sketch the graph of the function f