



Université d'Ottawa • University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Midterm 1 for MAT 1339B (Winter 2018) Introduction to Calculus and Vectors

Duration: 80 minutes

Professor: Rachid Bentoumi

NAME: _____

STUDENT NUMBER: _____

You must sign below.

Version 1

Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

- This is a closed book examination.
- Only Faculty standard calculators are permitted.
- There are 6 questions.
- The exam is out of 30 points.

Question 1. Calculate the following limits.

$$[2] \quad 1. \lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x - 3} = \left(\frac{0}{0}\right) \text{ I.F}$$

$$\lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x-2)}{(x-3)}$$

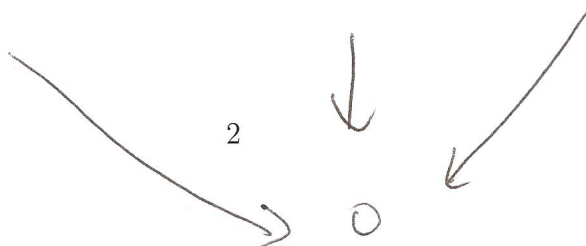
$$= \lim_{x \rightarrow 3} (x-2) = 3-2 = 1$$

$$[2] \quad 2. \lim_{x \rightarrow \infty} \frac{\cos(x)}{x^6 + 5}$$

$$-1 \leq \cos(x) \leq 1$$

$$\frac{-1}{x^6 + 5} \leq \frac{\cos(x)}{x^6 + 5} \leq \frac{1}{x^6 + 5}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{-1}{x^6 + 5} \leq \lim_{x \rightarrow \infty} \frac{\cos(x)}{x^6 + 5} \leq \lim_{x \rightarrow \infty} \frac{1}{x^6 + 5}$$



By Sandwich Theorem we have $\lim_{x \rightarrow \infty} \frac{\cos(x)}{x^6 + 5} = 0$

Question 2. Consider the following function

$$f(x) = \frac{x-3}{2-x^2}$$

[1] 1. Find the domain of f .

$$\begin{aligned} Df &= \left\{ x \in \mathbb{R} \mid 2-x^2 \neq 0 \right\} = \left\{ x \in \mathbb{R} \mid (\sqrt{2}-x)(\sqrt{2}+x) \neq 0 \right\} \\ &= \left\{ x \in \mathbb{R} \mid \sqrt{2}-x \neq 0 \text{ and } \sqrt{2}+x \neq 0 \right\} \\ &= \left\{ x \in \mathbb{R} \mid x \neq \sqrt{2} \text{ and } x \neq -\sqrt{2} \right\} \\ &= \mathbb{R} \setminus \left\{ -\sqrt{2}, \sqrt{2} \right\} \end{aligned}$$

[2] 2. Calculate the rate of change of f on $[-1, 1]$

$$\begin{aligned} \frac{f(1) - f(-1)}{1 - (-1)} &= \frac{\frac{1-3}{2-(1)^2} - \frac{-1-3}{2-(-1)^2}}{2} \\ &= \frac{\frac{-2}{1} - \frac{-4}{1}}{2} = \frac{-2+4}{2} = \frac{2}{2} = 1 \end{aligned}$$

Question 3. Let g be a function defined as: $g(x) = \sqrt{5-3x}$.

[3] 1. Use the definition of the derivative to find the derivative of g

$$\begin{aligned}
 g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{5-3(x+h)} - \sqrt{5-3x}}{h} \quad \left(\frac{0}{0}\right) \text{ I.F} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{5-3x-3h} - \sqrt{5-3x}}{h} \times \frac{\sqrt{5-3x-3h} + \sqrt{5-3x}}{\sqrt{5-3x-3h} + \sqrt{5-3x}} \\
 &= \lim_{h \rightarrow 0} \frac{(5-3x-3h) - (5-3x)}{h(\sqrt{5-3x-3h} - \sqrt{5-3x})} = \lim_{h \rightarrow 0} \frac{-3h}{h(\sqrt{5-3x-3h} + \sqrt{5-3x})} \\
 &= \frac{-3}{\sqrt{5-3x} + \sqrt{5-3x}} = \frac{-3}{2\sqrt{5-3x}}
 \end{aligned}$$

[2] 2. Find the tangent line to g at $x=0$.

$$\begin{aligned}
 y &= mx + b \quad \text{at } x=0 \Rightarrow g(0) = \sqrt{5-3(0)} = \sqrt{5} \\
 m &= g'(0) = \frac{-3}{2\sqrt{5-3(0)}} = \frac{-3}{2\sqrt{5}} = \frac{-3\sqrt{5}}{2\sqrt{5} \cdot \sqrt{5}} = -\frac{3\sqrt{5}}{10}
 \end{aligned}$$

$$\text{So, } y = \frac{-3}{2\sqrt{5}}x + b$$

at $x=0$, we have $y = g(0) = \sqrt{5}$

$$\Rightarrow \sqrt{5} = \frac{-3}{2\sqrt{5}}(0) + b \Rightarrow b = \sqrt{5}$$

$$\text{So, } y = -\frac{3}{2\sqrt{5}}x + \sqrt{5} = -\frac{3\sqrt{5}}{10}x + \sqrt{5}$$

Question 4. Consider the following function

$$f(x) = \begin{cases} 4 - x^2, & \text{if } x \leq 2 \\ kx - 2, & \text{if } x > 2 \end{cases}$$

[3] 1. Determine the value of k that makes the function f continuous on \mathbb{R} .

$$* \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 4 - x^2 = 4 - (2)^2 = 0$$

$$* \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} kx - 2 = 2k - 2$$

$$* f(2) = 4 - (2)^2 = 0$$

f is continuous $\forall x \in \mathbb{R}$. In particular, f is continuous at $x = 2$. $\Leftrightarrow \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$

$$\Leftrightarrow 0 = 2k - 2 = 0 \Rightarrow 2k = 2 \Rightarrow k = 1$$

[1] 2. Suppose that $k = -2$. Can we conclude that the derivative of f exists at $x = 2$? Justify your answer.

Method 1 From 1) f is continuous at $x = 2$ if $k = 1$. Now, if $k = -3$ then f is not continuous at $x = 2$. This means that f is not differentiable at $x = 2$. ($f'(2)$ does not exist)

Method 2 if $k = -2$ then,

$$f'(x) = \begin{cases} -2x, & x \leq 2 \\ -2, & x > 2 \end{cases} \Rightarrow f'(2) = \begin{cases} -4, & x \leq 2 \\ -2, & x > 2 \end{cases}$$

two different slopes $\Rightarrow f'(2)$ does not exist.

Question 5. Use the derivative rules to calculate the derivatives of the functions below.

[3] 1. $y = \sqrt{4x^2 + 7x - 1}$.

$$\begin{aligned} y' &= \left((4x^2 + 7x - 1)^{1/2} \right)' \\ &= \frac{1}{2} (4x^2 + 7x - 1)^{-1/2} (4x^2 + 7x - 1)' \\ &= \frac{1}{2} (8x + 7) (4x^2 + 7x - 1)^{-1/2} \\ &= \frac{8x + 7}{2\sqrt{4x^2 + 7x - 1}} \end{aligned}$$

[3] 2. $y = x^2(x^4 - 3)^3$.

$$\begin{aligned} y' &= (x^2)' (x^4 - 3)^3 + x^2 [(x^4 - 3)^3]' \\ &= 2x (x^4 - 3)^3 + x^2 (3) (x^4 - 3)^2 (x^4 - 3)' \\ &= 2x (x^4 - 3)^3 + 3x^2 (4x^3) (x^4 - 3)^2 \\ &= 2x (x^4 - 3)^3 + 12x^5 (x^4 - 3)^2 \end{aligned}$$

[3] 3. $y = \frac{x-1}{x^2+2}$.

$$\begin{aligned} y' &= \frac{(x-1)' (x^2+2) - (x-1) (x^2+2)'}{(x^2+2)^2} \\ &= \frac{(1) (x^2+2) - (x-1) (2x)}{(x^2+2)^2} \\ &= \frac{(x^2+2) - (x-1) (2x)}{(x^2+2)^2} \end{aligned}$$

Question 6. A manufacturer estimates that his weekly revenue $R(x)$ can be expressed by

$$R(x) = 1008x - 12x^2 - 8x^3$$

where x is the number of units sold, $R(x)$ is expressed in dollars. Under certain constraints, one cannot produce more than 10 units per week.

- [5] Use the **Second Derivative Test** to determine the quantity of units that gives a maximum revenue and evaluate this maximum revenue.

$$D_R = (0, 10]$$

critical points

$$R'(x) = (1008x - 12x^2 - 8x^3)' = 1008 - 24x - 24x^2$$

$$= -24(x^2 + x - 42)$$

$$R'(x) = 0 \Rightarrow x^2 + x - 42 = 0$$

$$\Delta = b^2 - 4ac = 1^2 - 4(42) = 169 = 13^2$$

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{-1 + \sqrt{13^2}}{2(1)} = \frac{13 - 1}{2} = 6 \in D_R$$

$$x_2 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{-1 - \sqrt{13^2}}{2(1)} = \frac{-1 - 13}{2} = -7 \notin D_R$$

$x_1 = 6 \in D_R$ is a c.p

In addition, $x = 10$ is also a c.p since $R'(10)$ does not exist.

* Second Derivative Test

$$R''(x) = (1008 - 24x - 24x^2)' = -24 - 48x$$

$$\text{at } \underline{x=6} \quad R''(6) = -24(6) - 48(6) = -312 < 0$$

$\Rightarrow x=6$ is a local max of $R(x)$ and the

$$\text{maximum is } R(6) = \$3888$$

Note that at $x=10$ $R''(10) = -504 < 0$

$x=10$ is also a local maximum and the

max is $R(10) = 880$. The abs max is then at $x=6$ and the abs max Revenue is $R(6) = 3888$.#