CARLETON UNIVERSITY

MATH 3705
FINAL
EXAMINATION
APRIL 2012

AUTHORIZED MEMORANDA
Non-programmable, non-graphic calculators

1. $\mathcal{L}\{e^{2t}\cos(3t)\} =$ (a) $\dfrac{s-2}{(s-2)^2+9}$ (b) $\dfrac{s+2}{(s+2)^2+9}$ (c) $\dfrac{s-3}{(s-3)^2+4}$ (d) $\dfrac{s-2}{s^2+9}$
   (e) None of these

2. $\mathcal{L}\{t\cos(2t)\} =$ (a) $\dfrac{4-s^2}{(s^2+4)^2}$ (b) $\dfrac{4s}{(s^2+4)^2}$ (c) $\dfrac{-4s}{(s^2+4)^2}$ (d) $\dfrac{s^2-4}{(s^2+4)^2}$
   (e) None of these

3. $\mathcal{L}^{-1}\left\{\dfrac{s+3}{s^2-6s+18}\right\} =$ (a) $e^{3t}[\cos(3t)+6\sin(3t)]$ (b) $e^{3t}[\cos(3t)+2\sin(3t)]$

   (c) $e^{3t}[\cos(3t)+\sin(3t)]$ (d) $e^{-3t}[\cos(2t)+2\sin(3t)]$ (e) None of these

4. $\mathcal{L}^{-1}\left\{\dfrac{(4s+7)e^{-2s}}{s^2+s-6}\right\} =$ (a) $u(t-2)\left[e^{-3t}+3e^{2t}\right]$ (b) $u(t-2)\left[e^{3t}+3e^{-2t}\right]$

   (c) $u(t-2)\left[e^{3(t-2)}+3e^{-2(t-2)}\right]$ (d) $u(t-2)\left[e^{-3(t-2)}+3e^{2(t-2)}\right]$ (e) None of these

5. If $y(t)$ denotes the solution of the initial-value problem

$$y'' - 2y' + 5y = 2\delta(t-2), \quad y(0) = 1, \quad y'(0) = 3,$$

   then $Y(s) = \mathcal{L}\{y(t)\} =$ (a) $\dfrac{s+3}{(s-1)^2+4}$ (b) $\dfrac{s+5+2e^{-2s}}{s^2-2s+5}$ (c) $\dfrac{s+1+2e^{-2s}}{s^2-2s+5}$
   (d) $\dfrac{s-1+2e^{-2s}}{(s-1)^2+4}$ (e) None of these

6. If $Y(s) = \mathcal{L}\{y(t)\} = \dfrac{2s+1}{s^2+s-2}$, then $y(t) =$ (a) $\dfrac{1}{3}(e^t - e^{-2t})$ (b) $e^{-t}+e^{2t}$
   (c) $2e^{-\frac{1}{2}t}\cos\left(\dfrac{3}{2}t\right)$ (d) $e^t + e^{-2t}$ (e) None of these

7. The general solution of the differential equation $4x^2y'' + 8xy' + y = 0$, valid for $x \neq 0$, is given by

   (a) $c_1|x|^{-\frac{1}{2}} + c_2|x|^{-\frac{1}{2}}$ (b) $|x|^{-\frac{1}{2}}(c_1 + c_2\ln|x|)$ (c) $c_1|x|^{-1+2\sqrt{3}} + c_2|x|^{-1-2\sqrt{3}}$

   (d) $|x|^{-1}[c_1\cos(2\sqrt{3}\ln|x|) + c_2\sin(2\sqrt{3}\ln|x|)]$ (e) None of these

8. The general solution of the differential equation $x^2y'' + 5xy' + 13y = 0$, valid for $x \neq 0$, is given by

(a) $|x|^{-\frac{5}{2}} \left[c_1 \cos\left(\frac{3\sqrt{3}}{2} \ln|x| \right) + c_2 \sin\left(\frac{3\sqrt{3}}{2} \ln|x| \right) \right]$ (b) $e^{-2x}[c_1 \cos(3x) + c_2 \sin(3x)]$

(c) $c_1|x| + c_2|x|^{-5}$ (d) $|x|^{-2} [c_1 \cos(3 \ln|x|) + c_2 \sin(3 \ln|x|)]$ (e) None of these

9. The coefficient recursion relation of the solution $y_1 = \sum_{n=0}^{\infty} a_n x^{n+1}$ of the differential equation $x^2y'' + (x^2 - x)y' + y = 0$ is

(a) $a_{n+1} = \frac{(n+1)a_n}{n^2}$ (b) $a_{n+1} = \frac{n^2 a_n}{n+1}$ (c) $a_{n+1} = \frac{a_n}{n+1}$ (d) $a_{n+1} = \frac{-a_n}{n+1}$

(e) None of these

10. The solution of the coefficient recursion relation $a_{n+1} = \frac{2a_n}{(n+1)^2}$, $n \geq 0$, is $a_n =$

(a) $\frac{2^n a_0}{(n!)^2}$ (b) $\frac{2a_0}{(n+1)^2}$ (c) $\frac{2^n a_0}{(n+1)!}$ (d) $\frac{2^n a_0}{[(n+1)!]^2}$ (e) None of these

11. One solution y_1 of the differential equation $x^2y'' + (x^2 + 3x)y' + y = 0$ has the form

(a) $\sum_{n=0}^{\infty} a_n x^n$ (b) $\sum_{n=0}^{\infty} a_n x^{n+1}$ (c) $\sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}$ (d) $\sum_{n=0}^{\infty} a_n x^{n-1}$ (e) None of these

12. The general solution of the differential equation $x^2y'' + xy' + (3x^2 - 4)y = 0$, valid for $x > 0$, is given by

(a) $c_1 J_2(\sqrt{3}\,x) + c_2 J_{-2}(\sqrt{3}\,x)$ (b) $c_1 J_{\sqrt{3}}(2x) + c_2 J_{-\sqrt{3}}(2x)$

(c) $c_1 J_2(\sqrt{3}\,x) + c_2 Y_2(\sqrt{3}\,x)$ (d) $c_1 J_{\sqrt{3}}(2x) + c_2 Y_{\sqrt{3}}(2x)$ (e) None of these

13. At $x = 59$, the Fourier sine series of $f(x) = \left\{ \begin{array}{ll} 2, & 0 \leq x < 1 \\ 4, & 1 \leq x \leq 3 \end{array} \right\}$ on $[0, 3]$ converges to

(a) 3 (b) -3 (c) 4 (d) -4 (e) -2

14. Let $f(x) = \left\{ \begin{array}{ll} x, & 0 \leq x \leq 1 \\ -x, & -1 \leq x \leq 0 \end{array} \right\}$, and $f(x+2) = f(x)$ for all x. The Fourier series of f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)],$$

where

(a) $a_0 = 1$, $a_n = \dfrac{2}{n^2\pi^2}[(-1)^n - 1]$, $b_n = \dfrac{-2(-1)^n}{n\pi}$, $n \geq 1$

(b) $a_0 = \dfrac{1}{2}$, $a_n = \dfrac{1}{n^2\pi^2}[(-1)^n - 1]$, $b_n = \dfrac{(-1)^{n-1}}{n\pi}$, $n \geq 1$

(c) $a_n = 0$, $n \geq 0$, $b_n = \dfrac{-2(-1)^n}{n\pi}$, $n \geq 1$

(d) $a_n = 0$, $n \geq 0$, $b_n = \dfrac{2(-1)^n}{n\pi}$, $n \geq 1$

(e) $a_0 = 1$, $a_n = \dfrac{2}{n^2\pi^2}[(-1)^n - 1]$, $b_n = 0$, $n \geq 1$

15. The solution of the heat equation $u_{xx} = u_t$, $0 < x < 1$, $t > 0$, which satisfies the boundary conditions $u(0,t) = u(1,t) = 0$ and the initial condition $u(x,0) = x$, is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t},$$

where $b_n =$

(a) $\dfrac{-2(-1)^n}{n\pi}$ (b) $\dfrac{2(-1)^n}{n\pi}$ (c) $\dfrac{2(-1)^n}{n^2\pi^2}$ (d) $\dfrac{-2(-1)^n}{n^2\pi^2}$ (e) None of these

16. The solution of the wave equation $u_{xx} = \dfrac{1}{4}u_{tt}$, $0 < x < 3$, $t > 0$, which satisfies the boundary conditions $u(0,t) = 0$ and $u(3,t) = 0$, and the initial conditions $u(x,0) = 0$ and $u_t(x,0) = 2\sin(\pi x) - 3\sin(2\pi x)$, is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{3}\right)\left[a_n \cos\left(\frac{2n\pi t}{3}\right) + b_n \sin\left(\frac{2n\pi t}{3}\right)\right],$$

where

(a) $a_1 = 2$, $a_2 = -3$, $a_n = 0$ otherwise, $b_n = 0$ for all $n \geq 1$

(b) $a_3 = 2$, $a_6 = -3$, $a_n = 0$ otherwise, $b_n = 0$ for all $n \geq 1$

(c) $b_1 = 2$, $b_2 = -3$, $b_n = 0$ otherwise, $a_n = 0$ for all $n \geq 1$

(d) $b_3 = 2$, $b_6 = -3$, $b_n = 0$ otherwise, $a_n = 0$ for all $n \geq 1$

(e) $b_3 = \dfrac{1}{\pi}$, $b_6 = \dfrac{-3}{4\pi}$, $b_n = 0$ otherwise, $a_n = 0$ for all $n \geq 1$

17. The solution $u(x,y)$ of Laplace's equation $u_{xx} + u_{yy} = 0$ within the rectangular region $0 < x < 3$, $0 < y < 2$, subject to the boundary conditions $u(0,y) = 0$, $u(3,y) = 0$,

$u(x,0) = 3x - x^2$, $u(x,2) = 0$, has the form

(a) $u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{3}\right) \sin\left(\frac{n\pi x}{3}\right)$

(b) $u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left[\frac{n\pi(2-y)}{3}\right] \sin\left(\frac{n\pi x}{3}\right)$

(c) $u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi x}{2}\right) \sin\left(\frac{n\pi y}{2}\right)$

(d) $u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left[\frac{n\pi(3-x)}{2}\right] \sin\left(\frac{n\pi y}{2}\right)$

(e) $u(x,y) = \alpha x + \beta y + \gamma xy + \delta$

18. The bounded solution of Laplace's equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = 3$, which satisfies the boundary condition $u(3,\theta) = 3 + 2\sin(2\theta) - \cos(3\theta)$, is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}[a_n \cos(n\theta) + b_n \sin(n\theta)],$$

where

(a) $a_0 = 6,\ a_3 = -27,\ b_2 = 18,\ a_n = b_n = 0$ otherwise

(b) $a_0 = 6,\ a_3 = -1,\ b_2 = 2,\ a_n = b_n = 0$ otherwise

(c) $a_0 = 6,\ a_3 = \frac{-1}{27},\ b_2 = \frac{2}{9},\ a_n = b_n = 0$ otherwise

(d) $a_0 = 6,\ a_2 = \frac{2}{9},\ b_3 = \frac{-1}{27},\ a_n = b_n = 0$ otherwise

(e) $a_0 = 6,\ a_2 = 18,\ b_3 = -27,\ a_n = b_n = 0$ otherwise

19. The differential equation $xy'' + 2y' + xy + \lambda xy = 0$, when placed in the Sturm-Liouville form $[p(x)y']' - q(x)y + \lambda r(x)y = 0$, has the weight function $r(x) =$

(a) 1 (b) x (c) x^2 (d) xe^{2x} (e) None of these

20. Given the Bessel identity $\frac{1}{\alpha}\frac{d}{dx}[x^\nu J_\nu(\alpha x)] = x^\nu J_{\nu-1}(\alpha x),\ \nu > 0,\ \alpha \neq 0,$

$\int_0^2 x^4 J_1(3x)\,dx =$ (a) $\frac{16}{9}[3J_2(6) - J_3(6)]$ (b) $16J_1(6)$ (c) $\frac{32}{5}J_2(6)$

(d) $16[J_2(6) - J_3(6)]$ (e) None of these

21. The eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0,\ 0 < x < 2, \quad y(0) = 0, \ \ y'(2) = 0,$$

are

(a) $\lambda_n = \frac{n\pi}{2}$, $y_n = B_n \sin\left(\frac{n\pi x}{2}\right)$, $n \geq 1$

(b) $\lambda_n = \frac{n\pi}{2}$, $y_n = A_n \cos\left(\frac{n\pi x}{2}\right)$, $n \geq 0$

(c) $\lambda_n = \frac{(2n+1)^2\pi^2}{16}$, $y_n = B_n \sin\left[\frac{(2n+1)\pi x}{4}\right]$, $n \geq 0$

(d) $\lambda_n = \frac{(2n+1)^2\pi^2}{16}$, $y_n = A_n \cos\left[\frac{(2n+1)\pi x}{4}\right]$, $n \geq 0$

(e) $\lambda_n = \frac{(2n+1)^2\pi^2}{4}$, $y_n = A_n \cos\left[\frac{(2n+1)\pi x}{2}\right]$, $n \geq 0$

22. $\mathcal{F}\left\{e^{-2ix-|x-3|}\right\} =$ (a) $\frac{2e^{-3i(\lambda-2)}}{1+(\lambda-2)^2}$ (b) $\frac{2e^{3i(\lambda-2)}}{1+(\lambda-2)^2}$ (c) $e\frac{2e^{-3i(\lambda+2)}}{1+(\lambda+2)^2}$

(d) $\frac{2e^{3i(\lambda+2)}}{1+(\lambda+2)^2}$ (e) None of these

23. $\mathcal{F}\left\{2xe^{-x^2}\right\} =$ (a) $i\sqrt{\pi}\lambda e^{-\frac{\lambda^2}{4}}$ (b) $2\lambda e^{-\lambda^2}$ (c) $\sqrt{\pi}e^{-\frac{\lambda^2}{4}}$ (d) $-i\sqrt{\pi}\lambda e^{-\frac{\lambda^2}{4}}$

(e) None of these

24. $\mathcal{F}^{-1}\left\{\frac{e^{-3i\lambda}}{1+(\lambda+2)^2}\right\} =$ (a) $\frac{1}{2}e^{2i(x+3)-|x+3|}$ (b) $\frac{1}{2}e^{-2i(x+3)-|x+3|}$

(c) $\frac{1}{2}e^{-2i(x-3)-|x-3|}$ (d) $\frac{1}{2}e^{2i(x-3)-|x-3|}$ (e) None of these

25. $\mathcal{F}^{-1}\left\{\lambda e^{-|\lambda|}\right\} =$ (a) $\frac{-i}{\pi(1+x^2)^2}$ (b) $\frac{i}{\pi(1+x^2)^2}$ (c) $\frac{-2ix}{\pi(1+x^2)^2}$

(d) $\frac{2ix}{\pi(1+x^2)^2}$ (e) None of these

Table of Laplace Transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ if } n \geq 0 \text{ is an integer}$$

$$\mathcal{L}\left\{t^p\right\} = \frac{\Gamma(p+1)}{s^{p+1}},\ p > -1$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

$$\mathcal{L}\{f(\alpha t)\} = \frac{1}{\alpha}F\left(\frac{s}{\alpha}\right), \quad \alpha > 0$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad s > a$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s), \quad s > a \geq 0$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0), \quad n \geq 0$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \equiv (-1)^n \frac{d^n}{ds^n}F(s), \quad n \geq 0$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(x)\,dx$$

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}F(s)$$

$$\mathcal{L}\{f(t)*g(t)\} \equiv \mathcal{L}\left\{\int_0^t f(t-x)g(x)\,dx\right\} = F(s)G(s), \text{ where } G(s) = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}, \quad a \geq 0$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt \text{ if } f \text{ is periodic with period } T$$

Summary of Fourier Series

1. The Fourier series of a $2L$-periodic function f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

with

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 0,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1,$$

where α is any real number. If f is an odd function, then

$$a_n = 0 \text{ and } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1.$$

If f is an even function, then

$$b_n = 0 \text{ and } a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 0.$$

2. The Fourier series of a function f defined on $[a, b]$ with $b - a = 2L$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

with

$$a_n = \frac{1}{L}\int_a^b f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 0,$$

$$b_n = \frac{1}{L}\int_a^b f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1.$$

If the $2L$-periodic extension $\tilde{f}$ of f to $\mathbb{R}$ is an odd function, then $a_n = 0$, and if $\tilde{f}$ is an even function, then $b_n = 0$.

3. The Fourier sine series of a function f defined on $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1.$$

4. The Fourier cosine series of a function f defined on $[0, L]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 0.$$

Table of Fourier Transforms

$$\mathcal{F}\big\{f(x)\big\} = \widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{i\lambda x}\, dx$$

$$\mathcal{F}^{-1}\big\{F(\lambda)\big\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{-i\lambda x}\, d\lambda$$

$$\mathcal{F}\big\{u(x-a) - u(x-b)\big\} = \frac{1}{i\lambda}\left(e^{i\lambda b} - e^{i\lambda a}\right),\ a < b$$

$$\mathcal{F}\big\{u(x+b) - u(x-b)\big\} = \frac{1}{i\lambda}\left(e^{i\lambda b} - e^{-i\lambda b}\right) = \frac{2}{\lambda}\sin(\lambda b)$$

$$\mathcal{F}\Big\{e^{-|x|}\Big\} = \frac{2}{1+\lambda^2}$$

$$\mathcal{F}\left\{e^{iax} f(x)\right\} = \widehat{f}(\lambda + a)$$

$$\mathcal{F}\big\{f(x-a)\big\} = e^{i\lambda a}\widehat{f}(\lambda)$$

$$\mathcal{F}\big\{f'(x)\big\} = -i\lambda\widehat{f}(\lambda)$$

$$\mathcal{F}\big\{xf(x)\big\} = -i\frac{d\widehat{f}}{d\lambda}$$

$$\mathcal{F}\left\{e^{-tx^2}\right\} = \sqrt{\frac{\pi}{t}}\, e^{-\frac{\lambda^2}{4t}},\ t > 0$$

$$\mathcal{F}\big\{f(\alpha x)\big\} = \frac{1}{|\alpha|}\widehat{f}\left(\frac{\lambda}{\alpha}\right),\ \alpha \neq 0$$

$$\mathcal{F}\big\{(f * g)(x)\big\} \equiv \mathcal{F}\left\{\int_{-\infty}^{\infty} f(s)g(x-s)\, ds\right\} = \widehat{f}(\lambda)\widehat{g}(\lambda), \text{ where } \widehat{g}(\lambda) = \mathcal{F}\big\{g(x)\big\}$$

$$\mathcal{F}\big\{\delta(x-a)\big\} = e^{i\lambda a}$$