Question 1. [4 points] Calculate

$$\text{a)} \quad \int_{-a}^{a} \frac{b}{y^2+y-12}\,dy \qquad \text{b)} \quad \int_{p}^{q} \frac{1}{x[1+(\ln x)^2]}\,dx$$

Solution. a) We have

$$\frac{b}{y^2+y-12} = \frac{b}{(y-3)(y+4)} = \frac{A}{y-3} + \frac{B}{y+4}$$

so

$$b = A(y+4) + B(y-3).$$

When $y = 3$, $A = b/7$ and when $y = -4$, $B = -b/7$. We thus have

$$\begin{aligned}\int_{-a}^{a} \frac{b}{y^2+y-12}\,dy &= \frac{b}{7}\left(\int_{-a}^{a} \frac{1}{y-3}\,dy - \int_{-a}^{a} \frac{1}{y+4}\,dy\right)\\ &= \frac{b}{7}\Big[\ln|y-3| - \ln|y+4|\Big]_{-a}^{a}\\ &= \frac{b}{7}\Big[\big(\ln|a-3| - \ln|a+4|\big) - \big(\ln|-a-3| - \ln|-a+4|\big)\Big]\end{aligned}$$

- **Version 1**: $a = 3$, $b = 14$, so the answer is

$$\begin{aligned}\frac{14}{7}\Big[(\ln|3-3| - \ln|3+4|) - (\ln|-3-3| - \ln|-3+4|)\Big] &= 2\Big[\big(\ln 0^+ - \ln 7\big) - (\ln 6 - 0)\Big]\\ &= -\infty\end{aligned}$$

- **Version 2**: $a = 2$, $b = 21$, so the answer is

$$\begin{aligned}\frac{21}{7}\Big[(\ln|2-3| - \ln|2+4|) - (\ln|-2-3| - \ln|-2+4|)\Big] &= 3\Big[(0 - \ln 6) - (\ln 5 - \ln 2)\Big]\\ &= -3\ln 15 = -8.124\end{aligned}$$

- **Version 3**: $a = 3$, $b = 28$, so the answer is

$$\begin{aligned}\frac{28}{7}\Big[(\ln|3-3| - \ln|3+4|) - (\ln|-3-3| - \ln|-3+4|)\Big] &= 4\Big[\big(\ln 0^+ - \ln 7\big) - (\ln 6 - 0)\Big]\\ &= -\infty\end{aligned}$$

b) Substituting $y = \ln x$, we have $\frac{dy}{dx} = \frac{1}{x}$, so $dx = x dy$.

$$\begin{aligned}\int_{p}^{q} \frac{1}{x(1+(\ln x)^2)}\,dx &= \int_{\ln p}^{\ln q} \frac{1}{1+y^2}\,dy\\ &= \int_{\ln p}^{\ln q} \frac{1}{1+y^2}\,dy\\ &= \Big[\arctan(y)\Big]_{\ln p}^{\ln q}\end{aligned}$$

- **Version 1**: $p = 1/e$ and $q = e$, so

$$\Big[\arctan(y)\Big]_{\ln 1/e}^{\ln e} = \Big[\arctan(y)\Big]_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2} = 1.570796$$

- **Version 2**: $p = 1/e$ and $q = 1$, so

$$\Big[\arctan(y)\Big]_{\ln 1/e}^{\ln 1} = \Big[\arctan(y)\Big]_{-1}^{0} = 0 - \left(-\frac{\pi}{4}\right) = \frac{\pi}{4} = 0.785398$$

- **Version 3**: $p = 1/e$ and $q = e$, so

$$\Big[\arctan(y)\Big]_{\ln 1/e}^{\ln e} = \Big[\arctan(y)\Big]_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2} = 1.570796$$

Question 2. [3 points] Solve the differential equation

$$\frac{dy}{dt} = \frac{at^2 \cos t}{y}$$

with initial condition $y(0) = b$.

Solution. First we separate the differential equation, putting the y's on the left and the t's on the right, and add an integral sign to obtain

$$\int y\,dy = \int at^2 \cos t\,dt.$$

The left-hand integral is simple: $y^2/2$. (We omit the $+C$ with this integral, since we'll include it with the other one when we're done integrating.)

The right-hand integral requires integration by parts twice. Set $u = t^2$, $v' = \cos t\,dt$, so that $u' = 2t$, $v = \sin t$. Then we obtain

$$\begin{aligned}\int at^2 \cos t\,dt &= a\int t^2 \cos t\,dt \\ &= a\left[t^2 \sin t - 2\int t \sin t\,dt\right]\end{aligned}$$

Using integration by parts again, with $u = t$, $v' = \sin t$, we have $u' = 1$, $v = -\cos t$. Then

$$\begin{aligned}\int at^2 \cos t\,dt &= a\left[t^2 \sin t - 2\left(-t\cos t + \int \cos t\right)\right] \\ &= a\left[t^2 \sin t + 2t\cos t - 2\sin t\right] + C\end{aligned}$$

So the general solution to the differential equation is

$$y^2/2 = a\left[t^2 \sin t + 2t\cos t - 2\sin t\right] + C;$$

solving for y gives

$$y = \pm\sqrt{2a(t^2 \sin t + 2t\cos t - 2\sin t) + 2C}. \tag{1}$$

This means that the various particular solutions are obtained by selecting one of ± and a particular constant C.

In our case, we're given the initial condition $y(0) = b$; that is, when t is 0, y takes the value b. Plugging in $t = 0$ into equation (1) gives

$$y(0) = \pm\sqrt{2a(0 + 0 - 0) + 2C} = \pm\sqrt{2C}$$

So $\pm\sqrt{2C} = b$. Clearly, to achieve the value of $b > 0$, we must select the positive square root. Squaring both sides gives $2C = b^2$. So the solution to our differential equation with initial condition is

$$y = \sqrt{2a(t^2 \sin t + 2t\cos t - 2\sin t) + b^2}.$$

(Note that the negative root is incorrect since $y(0) > 0$.)

- **Version 1**: $a = 7$, $b = 5$ so
$$y = \sqrt{14(t^2 \sin t + 2t\cos t - 2\sin t) + 25}.$$
- **Version 2**: $a = 3$, $b = 4$ so
$$y = \sqrt{6(t^2 \sin t + 2t\cos t - 2\sin t) + 16}.$$
- **Version 3**: $a = 2$, $b = 3$ so
$$y = \sqrt{4(t^2 \sin t + 2t\cos t - 2\sin t) + 9}.$$

Question 3. [4 points] Evaluate the integral

$$\int \frac{x^3 + 8x^2 + 5}{x^2 + 15x + 14}\, dx.$$

Solution. $P(x) = x^3+8x^2+5$ and $Q(x) = x^2+15x-14$. Then $\deg(P) = 3 > 2 = \deg(Q)$, so long division is necessary here. Using long division, we find that

$$x^3 + 8x^2 + 5 = (x - 7)(x^2 + 15x + 14) + (91x + 103)$$

or

$$\frac{x^3 + 8x^2 + 5}{x^2 + 15x + 14} = x - 7 + \frac{91x + 103}{x^2 + 15x + 14}$$

Notice that $x^2 + 15x + 14 = (x + 14)(x + 1)$. We need to find two numbers A and B so that

$$\frac{91x + 103}{(x + 14)(x + 1)} = \frac{A}{x + 14} + \frac{B}{x + 1}$$
$$91x + 103 = A(x + 1) + B(x + 14).$$

Substituting $x = -1$, we find $12 = 13B$, so $B = \frac{12}{13}$. Substituting $x = -14$, we find $-1171 = -13A$ so $A = \frac{1171}{13}$.

Finally,

$$\begin{aligned}\int \frac{x^3+8x^2+5}{x^2+15x+14}dx &= \int \left(x - 7 + \frac{1171}{13}\frac{1}{x+14} + \frac{12}{13}\frac{1}{x+1}\right)dx \\ &= \frac{1}{2}x^2 - 7x + \frac{1171}{13}\ln|x+7| + \frac{12}{13}\ln|x-2| + C.\end{aligned}$$

(Don't forget the absolute values or the $+C$.)

Question 4. [6 points] For each of the following improper integrals, determine whether it converges, and determine its value if it does.

a) $\displaystyle\int_1^a \frac{1}{t\,\ln t}\,dt$ b) $\displaystyle\int_0^\infty \frac{e^{bt}}{1+e^{2bt}}\,dt$ c) $\displaystyle\int_1^\infty \frac{\ln x}{x^n}\,dx$

Solution. a) Note that the function is not defined at 1, so we need to take a limit.

Using substitution, we have

$$\begin{aligned}u &= \ln t \\ \frac{du}{dt} &= \frac{1}{t} \\ dt &= tdu\end{aligned}$$

Then

$$\begin{aligned}\int_1^a \frac{dt}{t\ln t} &= \lim_{\epsilon\to1^+}\int_\epsilon^a \frac{dt}{t\ln t} \\ &= \lim_{\epsilon\to1^+}\int_{t=\epsilon}^{t=a}\frac{1}{u}du \\ &= \lim_{\epsilon\to1^+}\Big[\ln|u|\Big]_{t=\epsilon}^{t=a} \\ &= \lim_{\epsilon\to1^+}\Big[\ln|\ln t|\Big]_\epsilon^a \\ &= \lim_{\epsilon\to1^+}\ln|\ln a| - \ln|\ln\epsilon| \\ &= -\infty\end{aligned}$$

Thus, the integral diverges (regardless of the value of a).

b)

$$\int_0^\infty \frac{e^{bt}}{1+e^{2bt}}\,dt = \lim_{T\to\infty}\int_0^T \frac{e^{bt}}{1+e^{2bt}}\,dt$$

Let $u = e^{bt}$ so $dt = \frac{1}{be^{bt}} du$. Then

$$\begin{aligned}
\int_0^\infty \frac{e^{bt}}{1+e^{2bt}}\, dt &= \lim_{T\to\infty} \frac{1}{b} \int_{t=0}^{t=T} \frac{1}{1+u^2}\, dt \\
&= \frac{1}{b} \lim_{T\to\infty} \arctan u \Big|_{t=0}^{t=T} \\
&= \frac{1}{b} \lim_{T\to\infty} \arctan(e^{bt}) \Big|_0^T \\
&= \frac{1}{b} \lim_{T\to\infty} \arctan(e^{bT}) - \frac{1}{b} \arctan(1) \\
&= \frac{1}{b} \left(\frac{\pi}{2} - \frac{\pi}{4} \right) \\
&= \frac{\pi}{4b}
\end{aligned}$$

- **Version 1**: $b = 7$, so the answer is $\frac{\pi}{28} = 0.1121997$.
- **Version 2**: $b = 5$, so the answer is $\frac{\pi}{20} = 0.1570796$.
- **Version 3**: $b = 6$, so the answer is $\frac{\pi}{24} = 0.13089969$.

c)

$$\int_1^\infty \frac{\ln x}{x^n}\, dx = \lim_{T\to\infty} \int_1^T \frac{\ln x}{x^n}\, dx.$$

Using substitution, we have

$$\begin{aligned}
u &= \ln x \\
\frac{du}{dx} &= \frac{1}{x} \\
dx &= x du
\end{aligned}$$

Then

$$\int_1^\infty \frac{\ln x}{x^n}\, dx = \lim_{T\to\infty} \int_{t=1}^{t=T} \frac{u}{x^{n-1}}\, du$$

We need to resubstitute. Since $u = \ln x$, it follows that $x = e^u$. Thus

$$\int_1^\infty \frac{\ln x}{x^n}\, dx = \lim_{T\to\infty} \int_{t=1}^{t=T} u e^{(1-n)u}\, du.$$

Next we use integration by parts and let $w = u$ and $v' = e^{(1-n)u}$. Thus $w' = 1$ and $v = \frac{e^{(1-n)u}}{1-n}$. Then

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^n}\,dx &= \lim_{T\to\infty}\left[\frac{1}{1-n}ue^{(1-n)u} - \frac{1}{1-n}\int e^{(1-n)u}\,du\right] \\
&= \lim_{T\to\infty}\left[\frac{1}{1-n}ue^{(1-n)u} - \frac{1}{(1-n)^2}e^{(1-n)u}\right]_{x=1}^{x=T} \\
&= \lim_{T\to\infty}\left[\frac{1}{1-n}\ln x e^{(1-n)\ln x} - \frac{1}{(1-n)^2}e^{(1-n)\ln x}\right]_1^T \\
&= \lim_{T\to\infty}\left[\frac{1}{1-n}\frac{\ln x}{x^{n-1}} - \frac{1}{(1-n)^2x^{n-1}}\right]_1^T \\
&= \lim_{T\to\infty}\left[\frac{1}{1-n}\frac{\ln T}{T^{n-1}} - \frac{1}{(1-n)^2T^{n-1}}\right] - \left[-0 - \frac{1}{(n-1)^2}\right] \quad \text{(since } (1-n)^2 = (n-1)^2\text{)} \\
&= \lim_{T\to\infty}\left[\frac{1}{1-n}\frac{1/T}{(n-1)T^{n-2}} - \frac{1}{(1-n)^2T^{n-1}}\right] + \frac{1}{(n-1)^2} \quad \text{(using L'Hôpital's rule)} \\
&= \lim_{T\to\infty}\left[\frac{1}{-(n-1)^2T^{n-1}} - \frac{1}{(1-n)^2T^{n-1}}\right] + \frac{1}{(n-1)^2} \\
&= [0-0] + \frac{1}{(n-1)^2} \\
&= \frac{1}{(n-1)^2}
\end{aligned}$$

Therefore, the integral converges.

- **Version 1**: $n = 10$, so the answer is $\frac{1}{81} = 0.012345679$.
- **Version 2**: $n = 9$, so the answer is $\frac{1}{64} = 0.015625$.
- **Version 3**: $n = 8$, so the answer is $\frac{1}{49} = 0.020408$.

Question 5. [4 points] Find the volume of the solid obtained by rotating the area between $f(x) = 5x^2$ and $g(x) = -9x + 2$ about the x-axis.

Solution. The points of intersection of f and g satisfy

$$f(x) - g(x) = 5x^2 + 9x - 2 = (5x-1)(x+2) = 0.$$

Thus $x_1 = \frac{1}{5}$ and $x_2 = -2$ are the intersection points.

a) Within this interval, $g(x) > f(x)$, so the area is

$$\begin{aligned}
A &= \int_{-2}^{1/5} (-9x + 2 - 5x^2)dx \\
&= \left[-\frac{9x^2}{2} + 2x - \frac{5x^3}{3} \right]_{-2}^{1/5} \\
&= \left[\left(-\frac{9}{2(5^2)} + \frac{2}{5} - \frac{1}{3(5^2)} \right) - \left(-\frac{9(2^2)}{2} + 2(-2) - \frac{5(-2)^3}{3} \right) \right] \\
&= \frac{31}{150} - \left(-\frac{26}{3} \right) \\
&= \frac{1331}{150} = 8.87333333 \text{ units}^2.
\end{aligned}$$

b) The volume is

$$\begin{aligned}
V &= \pi \int_{-2}^{1/5} (-9x + 2)^2 - (5x^2)^2 dx \\
&= \pi \int_{-2}^{1/5} (81x^2 - 18x + 4 - 25x^4) dx \\
&= \pi \left[27x^3 - 9x^2 + 4x - 5x^5 \right]_{-2}^{1/5} \\
&= \pi \left[\left(\frac{27}{5^3} - \frac{9}{5^2} + \frac{4}{5} - \frac{5}{5^5} \right) - \left(27(-2)^3 - 9(-2)^2 + 4(-2) - 5(-2)^5 \right) \right] \\
&= \pi \left[0.6544 - (-100) \right] \\
&= \pi (100.6544) \\
&= 316.215 \text{ units}^3.
\end{aligned}$$

Question 6. [4 points] Zombies have invaded campus! Initially, there are 5 zombies. They recruit more of the undead to their ghoulish ranks at rate

$$\frac{dz}{dt} = at^2 e^{-bt},$$

where t is the time in days and z are the number of zombies. How many will be infected if the zombies recruited forever?

Solution. Using integration by parts, we have

$$\begin{aligned}
u &= at^2 & v' &= e^{-bt} \\
u' &= 2at & v &= -\frac{e^{-bt}}{b}
\end{aligned}$$

Thus

$$z = -\frac{at^2 e^{-bt}}{b} + \frac{2a}{b} \int t e^{-bt} dt.$$

Using integration by parts again, we have

$$u = t \qquad\qquad v' = e^{-bt}$$

$$u' = 1 \qquad\qquad v = -\frac{e^{-bt}}{b}$$

Hence

$$\begin{aligned} z &= -\frac{at^2e^{-bt}}{b} + \frac{2a}{b}\left(-\frac{te^{-bt}}{b} + \int \frac{e^{-bt}}{b}\right) \\ &= -\frac{at^2e^{-bt}}{b} - \frac{2ate^{-bt}}{b^2} - \frac{2ae^{-bt}}{b^3} + C \end{aligned}$$

Using the initial condition $z(0) = 5$, we have

$$z(0) = -0 - 0 - \frac{2a}{b^3}e^0 + C = -\frac{2a}{b^3} + C = 5\,.$$

Thus,

$$C = 5 + \frac{2a}{b^3}\,.$$

and hence

$$z(t) = -\frac{at^2e^{-bt}}{b} - \frac{2ate^{-bt}}{b^2} - \frac{2ae^{-bt}}{b^3} + 5 + \frac{2a}{b^3}\,.$$

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty}\left[-\frac{at^2e^{-bt}}{b} - \frac{2ate^{-bt}}{b^2} - \frac{2ae^{-bt}}{b^3} + 5 + \frac{2a}{b^3}\right]\,.$$

The last three terms aren't a problem, but the first two might be, because

$$\lim_{t\to\infty} -\frac{at^2e^{-bt}}{b} = \infty \times 0 \text{ and } \lim_{t\to\infty} -\frac{2ate^{-bt}}{b} = \infty \times 0$$

which we *cannot* evaluate as is. It's an indeterminant form and there's only one way to deal with those: L'Hôpital's rule. Which means we need to have it in a fractional form. Thus

$$\lim_{t\to\infty} -\frac{at^2e^{-bt}}{b} = \lim_{t\to\infty} -\frac{at}{be^{bt}} = \frac{\infty}{\infty} \quad\text{and}\quad \lim_{t\to\infty} -\frac{2ae^{-bt}}{b} = \lim_{t\to\infty} -\frac{2at}{be^{bt}} = \frac{\infty}{\infty}$$

Hence, we can apply L'Hôpital's rule twice in the first case and once in the second:

$$\begin{aligned} \lim_{t\to\infty} -\frac{at^2}{be^{bt}} = \lim_{t\to\infty} -\frac{2at}{b^2e^{bt}} = \lim_{t\to\infty} -\frac{2a}{b^3e^{bt}} = 0 \\ \text{and } \lim_{t\to\infty} -\frac{at}{be^{bt}} = \lim_{t\to\infty} -\frac{a}{b^2e^{bt}} = 0\,. \end{aligned}$$

We thus have

$$\begin{aligned} \lim_{t\to\infty} z(t) &= \lim_{t\to\infty}\left[-\frac{at^2e^{-bt}}{b} - \frac{2ate^{-bt}}{b^2} - \frac{2ae^{-bt}}{b^3} + 5 + \frac{2a}{b^3}\right] \\ &= 0 - 0 + 5 + \frac{2a}{b^3} \end{aligned}$$

- **Version 1**: $a = 8$ and $b = 0.07$ so the answer is $5 + \frac{16}{0.07^3} = 46652.23$, which means 46,652 zombies.
- **Version 2**: $a = 7$ and $b = 0.06$ so the answer is $5 + \frac{14}{0.06^3} = 64819.81$, which means 64,819 zombies.
- **Version 3**: $a = 12$ and $b = 0.09$ so the answer is $5 + \frac{24}{0.09^3} = 32926.81$, which means 32,926 zombies.

Question 7. [5 points] Determine the average value of $f(x) = x^n \ln(x)$ over the range $0 \leq x \leq e$.

Solution. The average value of $x^n \ln(x)$ over this interval is given by the quotient

$$\frac{\int_0^e x^n \ln(x)\,dx}{e-0} = \frac{1}{e}\int_0^e x^n \ln(x)\,dx. \tag{2}$$

This integral is improper because $\ln(x)$ is undefined at $x = 0$; that means it is given as a limit of definite integrals:

$$\int_0^e x^n \ln(x)\,dx = \lim_{\epsilon\to 0^+}\int_\epsilon^e x^n \ln(x)\,dx.$$

Using integration by parts, we have

$$\begin{aligned} u &= \ln x & v' &= x^n \\ u' &= \frac{1}{x} & v &= \frac{x^{n+1}}{n+1}. \end{aligned}$$

Thus,

$$\begin{aligned} &= \lim_{\epsilon\to 0^+}\left(\frac{x^{n+1}}{n+1}\ln x\Big|_\epsilon^e - \int_\epsilon^e \frac{x^n}{n+1}\,dx\right) \\ &= \lim_{\epsilon\to 0^+}\left(\frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2}\right)\Big|_\epsilon^e \\ &= \left(\frac{e^{n+1}}{n+1} - \frac{e^{n+1}}{(n+1)^2}\right) - \lim_{\epsilon\to 0^+}\left(\frac{\epsilon^{n+1}}{n+1}\ln\epsilon - \frac{\epsilon^{n+1}}{(n+1)^2}\right) \\ &= \left(\frac{ne^{n+1}}{(n+1)^2}\right) - \frac{1}{n+1}\lim_{\epsilon\to 0^+}\epsilon^{n+1}\ln\epsilon. \end{aligned}$$

Note that, in the last line, we recognised that as ϵ approaches 0, so will $\frac{\epsilon^{n+1}}{(n+1)^2}$.

Evaluating the remaining limit requires L'Hôpital's rule. We write the expression as a fraction in the most convenient way so that we can apply L'Hôpital:

$$\epsilon^{n+1}\ln\epsilon = \frac{\ln\epsilon}{\epsilon^{-n-1}}.$$

As ϵ approaches 0 (from the positive side), both numerator and denominator become infinite — the numerator approaches $-\infty$ and the denominator approaches ∞ — so the condition of L'Hôpital's rule is met and we can conclude that

$$\lim_{\epsilon\to 0^+} \frac{\ln\epsilon}{\epsilon^{-n-1}} = \lim_{\epsilon\to 0^+} \frac{1/\epsilon}{-(n+1)\epsilon^{-n-2}} = \lim_{\epsilon\to 0^+} -\frac{\epsilon^{n+1}}{n+1} = 0,$$

where we first applied L'Hôpital's rule and then simplified. So in the end, $\lim_{\epsilon\to 0^+} \epsilon^{n+1}\ln\epsilon = 0$, so the improper integral has value $\frac{ne^{n+1}}{(n+1)^2}$, and the average value of the function is given by equation (2) as

$$\frac{1}{e}\left(\frac{ne^{n+1}}{(n+1)^2}\right) = \frac{ne^n}{(n+1)^2}.$$

- **Version 1**: $n = 15$, so the answer is $\frac{15e^{15}}{256}$=191543.9866.
- **Version 2**: $n = 3$, so the answer is $\frac{3e^{3}}{16}$=3.7660.
- **Version 3**: $n = 7$, so the answer is $\frac{4e^{4}}{25}$=8.7357.