

MAT 1332, Fall 2017, Assignment 5

Due Wednesday November 22 in the math department dropboxes by 10:00pm.

Late assignments will not be accepted; nor will unstapled assignments.

Professors in the math department will not lend you a stapler; do not ask for one.

Name (Prime student) _____ Student Number _____
 Student Name _____ Student Number _____
 Student Name _____ Student Number _____

By signing below, we declare that this work is our own, that we have not copied from any other individual or other source and that all students contributed equally.

Signatures _____

QUESTION 1.

a) Solve the system of equations

$$\begin{aligned} 2x + 3y - z &= 8 \\ 4x - y &= 9 \\ -2x - 5y - 9z &= -10. \end{aligned}$$

We write the augmented matrix and the row reduce until it is upper triangular. Thus,

$$\begin{aligned} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 8 \\ 4 & -1 & 0 & 9 \\ -2 & -5 & -9 & -10 \end{array} \right] &\sim \left[\begin{array}{ccc|c} 2 & 3 & -1 & 8 \\ 0 & -7 & 2 & -7 \\ 0 & -2 & -10 & -2 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array} \\ &\sim \left[\begin{array}{ccc|c} 2 & 3 & -1 & 8 \\ 0 & -7 & 2 & -7 \\ 0 & 1 & 5 & 1 \end{array} \right] R_3 \rightarrow R_3 \div -2 \\ &\sim \left[\begin{array}{ccc|c} 2 & 3 & -1 & 8 \\ 0 & 0 & 37 & 0 \\ 0 & 1 & 5 & 1 \end{array} \right] R_2 \rightarrow R_2 + 7R_3 \\ &\sim \left[\begin{array}{ccc|c} 2 & 3 & -1 & 8 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 37 & 0 \end{array} \right] R_2 \leftrightarrow R_3. \end{aligned}$$

Since this is in upper triangular form, we can read off the equations (in reverse order):

$$\begin{aligned} 37z &= 0 &\Rightarrow z &= 0 \\ y + 5z &= 1 &\Rightarrow y &= 1 \\ 2x + 3y &= 8 \\ 2x &= 5 \\ x &= \frac{5}{2}. \end{aligned}$$

Hence, the solution is $(x, y, z) = (\frac{5}{2}, 1, 0)$.

b) Solve the system of equations

$$\begin{aligned} 3x - 4y + 3z &= 5 \\ 2x + 7y - 7z &= 1 \\ 13x + 2y - 5z &= 17. \end{aligned}$$

Once again, we write the augmented matrix and then row reduce. Thus,

$$\begin{aligned} \left[\begin{array}{ccc|c} 3 & -4 & 3 & 5 \\ 2 & 7 & -7 & 1 \\ 13 & 2 & -5 & 17 \end{array} \right] &\sim \left[\begin{array}{ccc|c} 1 & -\frac{4}{3} & 1 & \frac{5}{3} \\ 2 & 7 & -7 & 1 \\ 13 & 2 & -5 & 17 \end{array} \right] & R_1 \rightarrow R_1 \div 3 \\ &\sim \left[\begin{array}{ccc|c} 1 & -\frac{4}{3} & 1 & \frac{5}{3} \\ 0 & \frac{29}{3} & -9 & -\frac{7}{3} \\ 0 & \frac{58}{3} & -18 & \frac{14}{3} \end{array} \right] & \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 13R_1 \end{array} \\ &\sim \left[\begin{array}{ccc|c} 1 & -\frac{4}{3} & 1 & \frac{5}{3} \\ 0 & \frac{29}{3} & -9 & -\frac{7}{3} \\ 0 & 0 & 0 & 0 \end{array} \right] & R_3 \rightarrow R_3 - 2R_2. \end{aligned}$$

This is now in upper triangular form (even though there's a zero on the diagonal, there are only zeros below the diagonal, which is what we need). Since there's no z equation, we'll let

$$\begin{aligned} z &= t \\ \frac{29}{3}y - 9z &= -\frac{7}{3} \\ 29y - 27t &= -7 \\ y &= -\frac{7}{29} + \frac{27}{29}t \\ x - \frac{4}{3}y + z &= \frac{5}{3} \\ x - \frac{4}{3}\left(-\frac{7}{29} + \frac{27}{29}t\right) + t &= \frac{5}{3} \\ x &= \frac{39}{29} + \frac{7}{29}t. \end{aligned}$$

Thus, the solution is

$$(x, y, z) = \left(\frac{39}{29} + \frac{7}{29}t, -\frac{7}{29} + \frac{27}{29}t, t \right)$$

for any $t \in \mathbb{R}$.

c) Solve the system of equations

$$\begin{aligned}2x + y - z + 2w &= 3 \\6x - 2y + 3z - 10w &= 2 \\10x - 3y + 4z - 15w &= 2.\end{aligned}$$

(3 points) We have

$$\begin{aligned}\left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 3 \\ 6 & -2 & 3 & -10 & 2 \\ 10 & -3 & 4 & -15 & 2 \end{array} \right] &\sim \left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 3 \\ 0 & -5 & 6 & -16 & -7 \\ 0 & -8 & 9 & -25 & -13 \end{array} \right] &\begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 5R_1 \end{array} \\ &\sim \left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 3 \\ 0 & 1 & -\frac{6}{5} & -\frac{16}{5} & -\frac{7}{5} \\ 0 & -8 & 9 & -25 & -13 \end{array} \right] &R_2 \rightarrow R_2 \div -5 \\ &\sim \left[\begin{array}{cccc|c} 2 & 1 & -1 & 2 & 3 \\ 0 & 1 & -\frac{6}{5} & -\frac{16}{5} & -\frac{7}{5} \\ 0 & 0 & -\frac{33}{5} & \frac{3}{5} & -\frac{9}{5} \end{array} \right] &R_3 \rightarrow R_3 + 8R_2.\end{aligned}$$

This is upper triangular (because we have zeros below the main diagonal; that's still true even though we don't have a square matrix). Since we have more equations than unknowns, we need to let $w = t$. Then

$$\begin{aligned}-\frac{3}{5}z + \frac{3}{5}t &= -\frac{9}{5} \\z - t &= 3 \\z &= 3 + t \\y - \frac{6}{5}z + \frac{16}{5}w &= \frac{7}{5} \\y - \frac{18}{5} - \frac{6}{5}t + \frac{16}{5}t &= \frac{7}{5} \\y &= 5 - 2t \\2x + y - z + 2w &= 3 \\2x + 5 - 2t - (3 + t) + 2t &= 3 \\2x + 2 - t &= 3 \\2x &= 1 + t \\x &= \frac{1}{2} + \frac{1}{2}t.\end{aligned}$$

Thus, the solution is

$$(x, y, z, w) = \left(\frac{1}{2} + \frac{1}{2}t, 5 - 2t, 3 + t, t \right).$$

QUESTION 2.

a) Solve $2x^2 - 4x + 3 = 0$.

Using the quadratic formula, we have

$$\begin{aligned}x &= \frac{4 \pm \sqrt{16 - 4(2)(3)}}{4} \\&= \frac{4 \pm \sqrt{8}i}{4} \\&= 1 \pm \frac{1}{\sqrt{2}}i.\end{aligned}$$

b) Solve $3x^3 - 4x^2 - 2x + 5 = 0$.

(3 points) This one is a lot harder. In general, we can't solve cubics. But if we can guess a solution, then we can factor out. Let

$$f(x) = 3x^3 - 4x^2 - 2x + 5.$$

If we pick a few values of x , we see that

$$\begin{aligned}f(0) &= 5 \\f(1) &= 2 \\f(2) &= 9,\end{aligned}$$

so $x = 0, 1, 2$ aren't solutions. Let's try some negative values:

$$\begin{aligned}f(-1) &= 0 \\f(-2) &= -31.\end{aligned}$$

Since $f(-1) = 0$, it means $x = -1$ is a solution. So that means we can factor out, using long division. Thus

$$3x^3 - 4x^2 - 2x + 5 = (x + 1)(3x^2 - 7x + 5).$$

Using the quadratic formula on the second factor, we have

$$\begin{aligned}x &= \frac{7 \pm \sqrt{49 - 4(3)(5)}}{6} \\&= \frac{7}{6} \pm \frac{\sqrt{11}i}{6}.\end{aligned}$$

Thus, the three solutions are $x = -1, \frac{7}{6} + \frac{\sqrt{11}i}{6}, \frac{7}{6} - \frac{\sqrt{11}i}{6}$.

QUESTION 3. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Find A^2 .

We have

$$\begin{aligned} A^2 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

(b) Find A^3 .

Next, we have

$$\begin{aligned} A^3 &= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 3 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

(c) Find A^n for any positive integer n .

(2 points) In case you haven't seen the pattern yet, let's do one more:

$$\begin{aligned} A^4 &= \begin{pmatrix} 1 & 0 & 3 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 4 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Thus,

$$A^n = \begin{pmatrix} 1 & 0 & n \\ 0 & 2^n & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

QUESTION 4. Find the eigenvalues and the eigenvectors of

$$B = \begin{pmatrix} 4 & 3 & -4 \\ 0 & -2 & 0 \\ -7 & 6 & 1 \end{pmatrix}$$

We have

$$\begin{aligned} \det \begin{pmatrix} 4-\lambda & 3 & -4 \\ 0 & -2-\lambda & 0 \\ -7 & 6 & 1-\lambda \end{pmatrix} &= (4-\lambda)(-2-\lambda)(1-\lambda) + 0 + 0 - 28(-2-\lambda) - 0 - 0 \\ &= (-2-\lambda)[(4-\lambda)(1-\lambda) - 28] \\ &= (-2-\lambda)[\lambda^2 - 5\lambda - 24] \\ &= (-2-\lambda)(\lambda+3)(\lambda-8) = 0. \end{aligned}$$

Thus, the eigenvalues are $\lambda = -2, -3, 8$.

When $\lambda = -2$, we have

$$\begin{pmatrix} 4 - (-2) & 3 & -4 \\ 0 & -2 - (-2) & 0 \\ -7 & 6 & 1 - (-2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Equivalently, we could write this as an augmented matrix and row reduce. Thus,

$$\begin{aligned} \left(\begin{array}{ccc|c} 6 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ -7 & 6 & 3 & 0 \end{array} \right) &\sim \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ -7 & 6 & 3 & 0 \end{array} \right) & R_1 \rightarrow R_1 \div 6 \\ &\sim \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{19}{2} & -\frac{5}{3} & 0 \end{array} \right) & R_3 \rightarrow R_3 + 7R_1 \\ &\sim \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{2}{3} & 0 \\ 0 & \frac{19}{2} & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) & R_2 \leftrightarrow R_3. \end{aligned}$$

Thus, we need to let $x_3 = t$ (with $t \neq 0$ since $(0, 0, 0)^T$ isn't an eigenvector). Then we have

$$\begin{aligned} \frac{19}{2}x_2 &= \frac{5}{3}t \\ x_2 &= \frac{10}{57}t \\ x_1 + \frac{5}{57}t - \frac{2}{3}t &= 0 \\ x_1 &= \frac{11}{19}t. \end{aligned}$$

Thus, the eigenvectors corresponding to the eigenvalue $\lambda = -2$ are

$$\begin{pmatrix} \frac{11}{19} \\ \frac{10}{57} \\ 1 \end{pmatrix} t \quad t \in \mathbb{R}, t \neq 0.$$

When $\lambda = -3$, we have

$$\begin{aligned} \left(\begin{array}{ccc|c} 7 & 3 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ -7 & 6 & 4 & 0 \end{array} \right) &\sim \left(\begin{array}{ccc|c} 7 & 3 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 9 & 0 & 0 \end{array} \right) & R_3 \rightarrow R_3 + R_1 \\ &\sim \left(\begin{array}{ccc|c} 7 & 3 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) & R_3 \rightarrow R_3 - 9R_2. \end{aligned}$$

From the second equation, we have $x_2 = 0$, but we still have one equation and two variables, so we need to let $x_3 = s$ ($s \neq 0$). Then we have

$$\begin{aligned} 7x_1 + 3x_2 - 4x_3 &= 0 \\ 7x_1 + 3(0) - 4s &= 0 \\ x_1 &= \frac{4}{7}s. \end{aligned}$$

Thus, the eigenvectors corresponding to the eigenvalue $\lambda = -3$ are

$$\begin{pmatrix} \frac{4}{7} \\ 0 \\ 1 \end{pmatrix} s \quad s \in \mathbb{R}, s \neq 0.$$

When $\lambda = 8$, we have

$$\begin{aligned} \left(\begin{array}{ccc|c} -4 & 3 & -4 & 0 \\ 0 & -10 & 0 & 0 \\ -7 & 6 & -7 & 0 \end{array} \right) &\sim \left(\begin{array}{ccc|c} 1 & -\frac{3}{4} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -7 & 6 & -7 & 0 \end{array} \right) & \begin{array}{l} R_1 \rightarrow R_1 \div -4 \\ R_2 \rightarrow R_2 \div -10 \end{array} \\ &\sim \left(\begin{array}{ccc|c} 1 & -\frac{3}{4} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 \end{array} \right) & R_3 \rightarrow R_3 + 7R_1 \\ &\sim \left(\begin{array}{ccc|c} 1 & -\frac{3}{4} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) & R_3 \rightarrow R_3 - \frac{3}{4}R_2. \end{aligned}$$

We thus have $x_2 = 0$ and $x_3 = r$ ($r \neq 0$). Then

$$\begin{aligned} x_1 - \frac{3}{4}x_2 + x_3 &= 0 \\ x_1 &= -r. \end{aligned}$$

Thus, the eigenvectors corresponding to the eigenvalue $\lambda = 8$ are

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} r \quad r \in \mathbb{R}, r \neq 0.$$

QUESTION 5. Find the eigenvalues and the eigenvectors of

$$C = \begin{pmatrix} 5 & -17 \\ 1 & -3 \end{pmatrix}$$

(4 points) We have

$$\begin{aligned} \det \begin{pmatrix} 5 - \lambda & -17 \\ 1 & -3 - \lambda \end{pmatrix} &= (5 - \lambda)(-3 - \lambda) + 17 \\ &= \lambda^2 - 2\lambda + 2 = 0. \end{aligned}$$

Using the quadratic formula, we have

$$\begin{aligned} \lambda &= \frac{2 \pm \sqrt{4 - 4(2)}}{2} \\ &= 1 \pm i. \end{aligned}$$

When $\lambda = 1 + i$, we have

$$\begin{aligned} \left(\begin{array}{cc|c} 4 - i & -17 & 0 \\ 1 & -4 - i & 0 \end{array} \right) &\sim \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -4 - i & 0 \end{array} \right) \quad R_1 \rightarrow R_1 - (4 - i)R_2 \\ &\sim \left(\begin{array}{cc|c} 1 & -4 - i & 0 \\ 0 & 0 & 0 \end{array} \right) \quad R_1 \leftrightarrow R_2. \end{aligned}$$

Let $x_2 = t$ ($t \neq 0$). Then $x_1 = (4 + i)t$. Thus, the eigenvectors corresponding to the eigenvalue $\lambda = 1 + i$ are

$$\begin{pmatrix} 4 + i \\ 1 \end{pmatrix} t \quad t \in \mathbb{C}, t \neq 0.$$

When $\lambda = 1 - i$, we have

$$\begin{aligned} \left(\begin{array}{cc|c} 4 + i & -17 & 0 \\ 1 & -4 + i & 0 \end{array} \right) &\sim \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -4 + i & 0 \end{array} \right) \quad R_1 \rightarrow R_1 - (4 + i)R_2 \\ &\sim \left(\begin{array}{cc|c} 1 & -4 + i & 0 \\ 0 & 0 & 0 \end{array} \right) \quad R_1 \leftrightarrow R_2. \end{aligned}$$

Let $x_2 = s$ ($s \neq 0$). Then $x_1 = (4 - i)s$. Thus, the eigenvectors corresponding to the eigenvalue $\lambda = 1 - i$ are

$$\begin{pmatrix} 4 - i \\ 1 \end{pmatrix} s \quad s \in \mathbb{C}, s \neq 0.$$