MAT 1332, Fall 2017, Assignment 4

Due Wednesday November 8 in the math department dropboxes by 10:00pm.

Late assignments will not be accepted; nor will unstapled assignments.

Professors in the math department will not lend you a stapler; do not ask for one.

Please print double sided to save paper.

Name (Prime student) ____________________ Student Number ____________________

Student Name ____________________ Student Number ____________________

Student Name ____________________ Student Number ____________________

By signing below, we declare that this work is our own, that we have not copied from any other individual or other source and that all students contributed equally.

Signatures ____

QUESTION 1. Consider the autonomous differential equation

$$\frac{dx}{dt} = f(x) = x^2 - 5x + 6, \qquad x(0) = 0.$$

(a) Find the steady states x_1^*, x_2^*.

To find the steady states, we write down the equation for the equilibria,

$$x^2 - 5x + 6 = 0$$
$$(x-2)(x-3) = 0$$

Thus,

$$x - 2 = 0 \quad \text{or} \quad x - 3 = 0 \quad \text{(set each factor equal to 0)}$$
$$x = 2 \quad \text{or} \quad x = 3 \quad \text{(solve each equation)}$$

So the steady states are $x_1^* = 2$ and $x_2^* = 3$.

(b) Use the stability test to find the stability of x_1^*, x_2^*.

(2 points) The derivative of the rate of change $f(x)$ with respect to x is

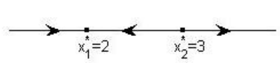
$$f'(x) = 2x - 5,$$

Then

$$f'(x_1^*) = f'(2) = 2 \cdot 2 - 5 = -1 < 0$$
$$f'(x_2^*) = f'(3) = 2 \cdot 3 - 5 = 1 > 0.$$

Hence, $x_1^* = 2$ is stable and $x_2^* = 3$ is unstable.

(c) Draw the phase-line diagram.



(1 point)

(d) Use separation of variables to solve the equation explicitly.

Factoring, we have $x^2 - 5x + 6 = (x-2)(x-3)$. Separating variables gives

$$\frac{1}{(x-2)(x-3)}dx = dt$$
$$\int \frac{1}{(x-2)(x-3)}dx = \int dt$$

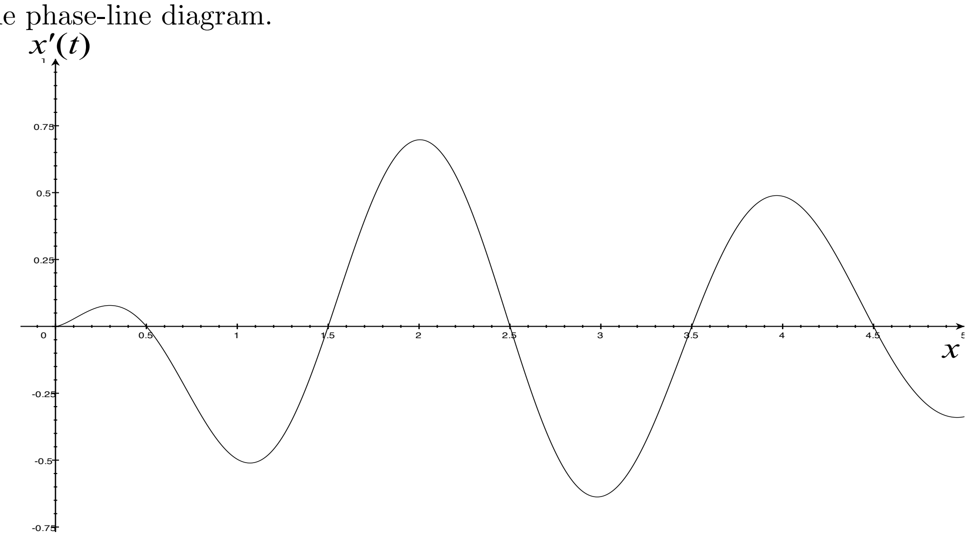
Using partial fractions, we have

$$\frac{1}{(x-2)(x-3)}dx = \frac{A}{x-2} + \frac{B}{x-3}$$
$$1 = A(x-3) + B(x-2)$$
$$x = 2: \quad 1 = A(-1) \quad A = -1$$
$$x = 3: \quad 1 = B(-1) \quad B = -1$$

The integral is thus

$$\int \left(-\frac{1}{x-2} - \frac{1}{x-3}\right) dx = \int dt$$
$$-\ln|x-2| - \ln|x-3| = t + c$$
$$\ln|(x-2)(x-3)| = -t - c$$
$$x^2 - 5x + 6 = e^{-t-c}$$
$$x^2 - 5x + 6 - e^{-t-c} = 0$$
$$x = \frac{5 \pm \sqrt{25 - 4(6 - e^{-t-c})}}{2}$$

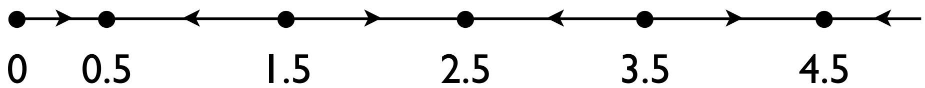
QUESTION 2. From the following graph of the rate of change as a function of the state variable, (a) identify all critical points, (b) classify each critical point as stable or unstable, and (c) draw the phase-line diagram.



(2 points)

The critical points are $x = 0$, 0.5, 1.5, 2.5, 3.5 and 4.5.

A critical point is stable if the slope is negative at that point. Thus, $x = 0.5$, 2.5 and 4.5 are stable, while $x = 0$, 1.5 and 3.5 are unstable.



Question 3. Consider a disease that is transmitted both horizontally and vertically. People can be either susceptible or infected. Infection occurs either from susceptibles being infected or from infected mothers giving birth to infected children. Susceptible mothers will, of course, give birth to susceptible children. People can also recover from this disease, which does not kill. Write down the differential equations that describe this outbreak. (Let the horizontal transmission parameter be α, the birth rate be b and the recovery rate be μ.) Write down all equilibria of this model. Under what conditions on the parameters will this disease be endemic?

The equations are

$$\frac{dS}{dt} = bS - \alpha IS + \mu I$$
$$\frac{dI}{dt} = bI + \alpha IS - \mu I$$

Since $S + I = N$, we can write the second equation as

$$\frac{dI}{dt} = bI + \alpha I(N - I) - \mu I$$

Solving $I' = 0$, we have

$$0 = I(b + \alpha N - \alpha I - \mu)$$

Hence $I = 0$ or $I = \frac{b+\alpha N-\mu}{\alpha}$. So the equilibria are

$$(\bar{S}, \bar{I}) = (N, 0), \left(N - \frac{b + \alpha N - \mu}{\alpha}, \frac{b + \alpha N - \mu}{\alpha}\right)$$

The second equilibrium only exists if $\bar{I} > 0$. That is, $b + \alpha N - \mu > 0$.

Note: the choice of equations is not unique. For example, there might be a separate equation for recovered individuals.

Question 4.

a) Express $z_1 = e^{i\pi/2}$ and $z_2 = e^{i\pi/6}$ in the form $z = a + ib$.

$$z_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$$
$$z_2 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

b) Express z_1/z_2 in the form $z = a + ib$.

$$\frac{z_1}{z_2} = \frac{e^{i\pi/2}}{e^{i\pi/6}} = e^{i(\pi/2-\pi/6)} = e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

c) Express $\omega = -12 + 5i$ in the form $\omega = re^{i\theta}$.

(1 point) We have $r = \sqrt{(-12)^2 + 5^2} = 13$. Since the vector is in the third quadrant, we have

$$\theta = \pi - \arctan\frac{5}{12} = 2.7468.$$

Thus,

$$\omega = 13e^{2.7468i}$$

d) Find $\omega\bar{\omega}$.

We have $\bar{\omega} = 13e^{-2.7468i}$, so

$$\omega\bar{\omega} = 13e^{2.7468i}13e^{-2.7468i} = 169.$$

Alternatively, we could just have multiplied out:

$$\omega\bar{\omega} = (-12 + 5i)(-12 - 5i) = 144 - 25i^2 = 169.$$