MAT 1332, Fall 2017, Assignment 3
Due Wednesday October 11 in the math department dropboxes by 10:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.
Please print double sided to save paper.

Name (Prime student) ______________________ Student Number ____________________
Student Name ______________________________ Student Number ____________________
Student Name ______________________________ Student Number ____________________

By signing below, we declare that this work is our own, that we have not copied from any other individual or other source and that all students contributed equally.

Signatures ____

QUESTION 1. Determine whether the following integrals converge or diverge. If they converge, calculate the value.

a) $\displaystyle\int_1^\infty \frac{1}{\sqrt[5]{x}}dx$ b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cot\theta}{\csc\theta}d\theta$ c) $\displaystyle\int_1^\infty \frac{\arctan(x)}{1+x^2}dx$ d) $\displaystyle\int_1^7 \ln\left(\frac{e^t}{t^3}\right)dt$

(a)
Using the definition of an improper integral, we have

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt[5]{x}}dx &= \lim_{T\to\infty}\int_1^t x^{-1/5}dx\\ &= \lim_{T\to\infty}\left.\frac{5x^{4/5}}{4}\right|_1^t\\ &= \lim_{T\to\infty}\left[\frac{5T^{4/5}}{4}-\frac{5}{4}\right]\\ &= \infty\,.\end{aligned}$$

Thus, this integral diverges.

(b)
Since $\cot 0$ and $\csc 0$ are both undefined, this is an improper integral. Hence

$$\int_0^{\frac{\pi}{2}} \frac{\cot\theta}{\csc\theta}d\theta = \lim_{T\to 0}\int_T^{\frac{\pi}{2}} \frac{\cot\theta}{\csc\theta}d\theta\,.$$

Since $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$ and $\csc\theta = \frac{1}{\sin\theta}$, we have

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\cot\theta}{\csc\theta} d\theta &= \lim_{T\to 0} \int_T^{\frac{\pi}{2}} \frac{\cos\theta/\sin\theta}{1/\sin\theta} d\theta \\
&= \lim_{T\to 0} \int_T^{\frac{\pi}{2}} \cos\theta d\theta \\
&= \lim_{T\to 0} \sin\theta \Big|_T^{\frac{\pi}{2}} \\
&= \lim_{T\to 0} \left[\sin\frac{\pi}{2} - \sin T\right] \\
&= 1\,.
\end{aligned}$$

(c)

(2 points) We have

$$\int_1^{\infty} \frac{\arctan(x)}{1+x^2} dx = \lim_{T\to\infty} \int_1^T \frac{\arctan(x)}{1+x^2} dx\,.$$

Using integration by parts, we have

$$\begin{aligned}
u &= \arctan x & v' &= \frac{1}{1+x^2} \\
u' &= \frac{1}{1+x^2} & v &= \arctan x\,.
\end{aligned}$$

Hence,

$$\lim_{T\to\infty} \int_1^T \frac{\arctan(x)}{1+x^2} dx = \lim_{T\to\infty} (\arctan x)^2 \Big|_0^T - \lim_{T\to\infty} \int_1^T \frac{\arctan x}{1+x^2} dx\,.$$

However, the second integral is just the first except for a minus sign. Thus, if we add it to both sides, then

$$2 \lim_{T\to\infty} \int_1^T \frac{\arctan(x)}{1+x^2} dx = \lim_{T\to\infty} (\arctan x)^2 \Big|_0^T\,.$$

Thus,

$$\begin{aligned}
\lim_{T\to\infty} \int_1^T \frac{\arctan(x)}{1+x^2} dx &= \frac{1}{2} \lim_{T\to\infty} (\arctan x)^2 \Big|_1^T \\
&= \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right] \\
&= \frac{3\pi^2}{32} = 0.925275
\end{aligned}$$

(1 point for setting up integration by parts, 0.5 points for the creating a single integral, 0.5 for the answer.)

(d)

This isn't an improper integral at all, since there are no domain problems in $1 \leq t \leq 7$. So it probably converges.

Using log laws, we have

$$\begin{aligned} \int_1^7 \ln\left(\frac{e^t}{t^3}\right) dt &= \int_1^7 (\ln e^t - \ln t^3) dt \\ &= \int_1^7 (t - 3\ln t) dt \\ &= \frac{t^2}{2}\bigg|_1^7 - \int_1^7 3\ln t dt\,. \end{aligned}$$

Using integration by parts, we have

$$\begin{aligned} u &= \ln t & v' &= 3 \\ u' &= \frac{1}{t} & v &= 3t\,. \end{aligned}$$

Thus,

$$\begin{aligned} \int_1^7 (t - 3\ln t) dt &= \left[\frac{t^2}{2} - 3t\ln t\right]_1^7 + \int 3dt \\ &= \left[\frac{t^2}{2} - 3t\ln t + 3t\right]_1^7 \\ &= 42 - 21\ln 7 = 1.13588\,. \end{aligned}$$

QUESTION 2. Evaluate the following indefinite integrals:

$$\text{a)} \int \frac{x^3 + x^2 + 5}{x^2 + 4} dx \qquad\qquad \text{b)} \int \frac{x^4 - 8x^2 - 10}{x^2 - 3x - 10} dx$$

(a)

(3 points) For the integral $I = \int \frac{x^3+x^2+5}{x^2+4} dx$, the numerator $x^3 + x^2 + 5$ has higher degree than the denominator $x^2 + 4$, so we need to use long division. If we do this, the result is $x + 1$ with remainder $-4x + 1$. Thus,

$$\frac{x^3 + x^2 + 5}{x^2 + 4} = x + 1 + \frac{-4x + 1}{x^2 + 4} = x + 1 - \frac{4x}{x^2 + 4} dx + \frac{1}{x^2 + 4}$$

so we can calculate

$$I = \int (x + 1) dx - \int \frac{4x}{x^2 + 4} dx + \int \frac{1}{x^2 + 4} dx\,.$$

The first integral is easy. For the second, we'll use the substitution $u = x^2 + 4$. For the third, we'll rearrange and use arctan. Thus,

$$I = \int (x + 1) dx - \int \frac{4x}{x^2 + 4} dx + \frac{1}{4} \int \frac{1}{\left(\frac{x}{2}\right)^2 + 1} dx\,.$$

Now we substitute $w = \frac{x}{2}$ in the last integral. Then

$$\begin{aligned} I &= \int (x+1)dx - \int \frac{4x}{u}\frac{du}{2x} + \frac{1}{4}\int \frac{1}{w^2+1}2dw \\ &= \frac{x^2}{2} + x - 2\int \frac{1}{u}du + \frac{1}{2}\int \frac{1}{w^2+1}dw \\ &= \frac{x^2}{2} + x - 2\ln|u| + \frac{1}{2}\arctan w + C \\ &= \frac{x^2}{2} + x - 2\ln(x^2+4) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C\,. \end{aligned}$$

Remarks. 1. Notice that we can drop the absolute value signs because x^2+4 is guaranteed to be positive.

2. Because of the integration constant, there may be some variations on this solution, due to the logarithm. For example,

$$\frac{x^2}{2} + x - 2\ln\left[\left(\frac{x}{2}\right)^2 + 1\right] + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \tilde{C}$$

is an equivalent solution.

(0.5 for each term in the final answer; no part marks)

(b)

Once again, we need to use long division. This gives us

$$\frac{x^4 - 8x^2 - 10}{x^2 - 3x - 10} = x^2 + 3x + 11 + \frac{63x + 100}{x^2 - 3x - 10}\,.$$

The first part is easy, the second is going to require more work. Let's factorise the denominator:

$$\frac{63x+100}{x^2-3x-10} = \frac{63x+100}{(x-5)(x+2)}\,.$$

Then we can use partial fractions:

$$\frac{63x+100}{(x-5)(x+2)} = \frac{A}{x-5} + \frac{B}{x+2}\,.$$

If we multiply both sides by $(x-5)(x+2)$, then

$$63x + 100 = A(x+2) + B(x-5)\,.$$

Substituting $x = 5$ gives

$$415 = 7A \;\Rightarrow A = \frac{415}{7}\,.$$

Substituting $x = -2$ gives

$$-26 = -7B \Rightarrow B = \frac{26}{7}\,.$$

Thus

$$\frac{63x+100}{(x-5)(x+2)} = \frac{415}{7}\frac{1}{x-5} + \frac{26}{7}\frac{1}{x+2}\,.$$

Our integral is thus

$$\begin{aligned}\frac{x^4-8x^2-10}{x^2-3x-10} &= \int\left(x^2+3x+11+\frac{415}{7}\frac{1}{x-5}+\frac{26}{7}\frac{1}{x+2}\right)dx \\ &= \frac{x^3}{3}+\frac{3x^2}{2}+11x+\frac{415}{7}\ln|x-5|+\frac{26}{7}\ln|x+2|+C\,.\end{aligned}$$

Don't forget the absolute value signs on the logarithms; you definitely need them!

Question 3. Show that

$$x(t) = -\frac{1}{2}\left(1-5e^{6t}\right)$$

is a solution of the differential equation

$$\frac{dx}{dt} = 3+6x$$

which satisfies the initial condition $x(0)=2$.

First we verify that $x(t)$ is a solution of the differential equation. To do this, we substitute the expression into both the left- and right-hand sides of the differential equation.

$$\begin{aligned}\text{LHS} &= \frac{dx}{dt} = \frac{1}{2}\left(6\cdot 5e^{6t}\right) = 15e^{6t} \\ \text{RHS} &= 3+6x = 3-\frac{6}{2}\left(1-5e^{6t}\right) = 15e^{6t}\,.\end{aligned}$$

Thus, since the left-hand side equals the right-hand side, the solution satisfies the equation. Next we check that it satisfies the initial condition.

$$x(0) = -\frac{1}{2}\left(1-5e^{0}\right) = -\frac{1}{2}(1-5) = 2\,.$$

Thus, $x(t)$ is a solution of the differential equation.

(Don't forget that we have to do two things here!)

Question 4. The rate of change of a wolf population is described by

$$\frac{dP}{dt} = 500\ln(1+\alpha)\times(1+\alpha)^{-t}\,,$$

where $P(t)$ is the number of wolves as a function of time, t is measured in years and $\alpha=0.03$ is the growth rate parameter.

a) In five years' time, will the population be larger or smaller than it is now? Larger

b) Assuming $\alpha>0$, does the answer depend on the size of α? No.

c) Could there be a different answer to (b) if $\alpha<0$? Yes.

(1 mark for each answer above. No part marks.)

The big challenge here is dealing with $(1+\alpha)^{-t}$. How do we cope with this? Answer: that's why they invented logarithms! Remember, logarithms turn powers into products. But we can't just take a logarithm in isolation, we need to balance it by taking an exponential as well (since they're inverses).

There's a second issue that's less obvious and that's the lack of initial condition. But remember that every differential equation has an initial condition, whether it's explicitly there or not. So we can assume that initially (ie at time $t=0$) the wolf population is at some level $P(0)$.

Since we'll need to think about several different values of α, let's not substitute that just yet.

Solving the differential equation leads us to

$$\begin{aligned}
P(t) &= P(0) + 500\ln(1+\alpha)\int_0^t (1+\alpha)^{-s}ds \\
&= P(0) + 500\ln(1+\alpha)\int_0^t e^{-s\ln(1+\alpha)}ds \qquad (\text{since } e^{\ln(1+\alpha)^{-s}} = (1+\alpha)^{-s}) \\
&= P(0) + 500\ln(1+\alpha)\left[\frac{e^{-s\ln(1+\alpha)}}{-\ln(1+\alpha)}\right]_0^t \\
&= P(0) - 500\left[e^{-s\ln(1+\alpha)}\right]_0^t \\
&= P(0) - 500e^{-t\ln(1+\alpha)} + 500 \\
&= P(0) - 500(1+\alpha)^{-t} + 500\,.
\end{aligned}$$

We're now in a position to answer our questions.

a) Regardless of what $P(0)$ actually is, when $\alpha = 0.03$, we have

$$P(5) = P(0) + 500(1-(1.03)^{-5}) = P(0) + 68.6956\,.$$

Thus, the population has definitely increased after five years.

b) If we didn't know α, then

$$\begin{aligned}
P(5) &= P(0) + 500\left(1-(1+\alpha)^{-5}\right) \\
&= P(0) + 500\left(1 - \frac{1}{(1+\alpha)^5}\right) \\
&= P(0) + 500\left(\frac{(1+\alpha)^5 - 1}{(1+\alpha)^5}\right) \\
&= P(0) + 500\left(\frac{1+5\alpha+10\alpha^2+10\alpha^3+5\alpha^4+\alpha^5-1}{(1+\alpha)^5}\right) \\
&= P(0) + 500\left(\frac{5\alpha+10\alpha^2+10\alpha^3+5\alpha^4+\alpha^5}{(1+\alpha)^5}\right) \\
&> P(0)
\end{aligned}$$

if $\alpha > 0$.

c) If $\alpha < 0$, say $\alpha = -0.03$, then

$$P(5) = P(0) + 500\left(1 - (0.97)^{-5}\right) = P(0) - 82.252\,.$$

It follows that the population could become smaller if $\alpha < 0$.

(That's not guaranteed though. What happens if $\alpha = -1$ or $\alpha = -2$?)