MAT 1332, Fall 2017, Assignment 2
Due Wednesday October 4 in the math department dropboxes by 10:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.
Please print double sided to save paper.

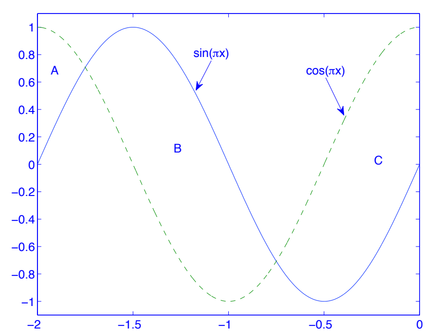
Name (Prime student) ____________________ Student Number ____________________
Student Name ____________________ Student Number ____________________
Student Name ____________________ Student Number ____________________

By signing below, we declare that this work is our own, that we have not copied from any other individual or other source and that all students contributed equally.

Signatures ____

QUESTION 1. Find the area between $f(x) = \sin(\pi x)$ and $g(x) = \cos(\pi x)$ for $-2 < x < 0$. (Hint: Sketch the curves.)

If you simply integrate, the answer is zero. So that obviously isn't right. It's best to sketch the two graphs.



Clearly the area between them isn't zero. In fact, it's area A, plus area B, plus area C on the graph. And before we can find them, we need to find the points of intersection. That is, the values of x where $\sin(\pi x) = \cos(\pi x)$. Let's figure them out:

$$\begin{aligned}
\sin(\pi x) &= \cos(\pi x) \\
\tan(\pi x) &= 1 && \text{(dividing both sides by } \cos(\pi x)\text{)} \\
\pi x &= \ldots -\frac{9\pi}{4}, -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4} \ldots \\
x &= \ldots -\frac{9}{4}, -\frac{7}{4}, -\frac{3}{4}, \frac{1}{4}, \frac{5}{4}, \frac{9}{4} \ldots
\end{aligned}$$

Actually, since $-2 \le x \le 0$, we only need values that fall into this range. So the points of intersection within this range are $x = -\frac{7}{4}$ and $x = -\frac{3}{4}$.

Now we're ready to integrate. The area is the sum of areas A, B and C, so we have

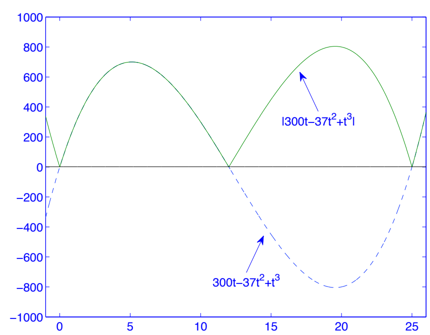
$$
\begin{aligned}
\text{Area} &= A + B + C \\
&= \int_{-2}^{-7/4} (\cos(\pi x) - \sin(\pi x))\, dx + \int_{-7/4}^{-3/4} (\sin(\pi x) - \cos(\pi x))\, dx + \int_{-3/4}^{0} (\cos(\pi x) - \sin(\pi x))\, dx \\
&= \left[\frac{\sin(\pi x)}{\pi} + \frac{\cos(\pi x)}{\pi}\right]_{-2}^{-7/4} + \left[-\frac{\cos(\pi x)}{\pi} - \frac{\sin(\pi x)}{\pi}\right]_{-7/4}^{-3/4} + \left[\frac{\sin(\pi x)}{\pi} + \frac{\cos(\pi x)}{\pi}\right]_{-3/4}^{0} \\
&= \left[\left(\frac{\sin(-7\pi/4)}{\pi} + \frac{\cos(-7\pi/4)}{\pi}\right) - \left(\frac{\sin(-2\pi)}{\pi} + \frac{\cos(-2\pi)}{\pi}\right)\right] + \left[\left(-\frac{\cos(-3\pi/4)}{\pi} - \frac{\sin(-3\pi/4)}{\pi}\right)\right. \\
&\quad \left. - \left(-\frac{\cos(-7\pi/4)}{\pi} - \frac{\sin(-7\pi/4)}{\pi}\right)\right] + \left[\left(\frac{\sin(0)}{\pi} + \frac{\cos(0)}{\pi}\right) - \left(\frac{\sin(-3\pi/4)}{\pi} + \frac{\cos(-3\pi/4)}{\pi}\right)\right] \\
&= \left[\left(\frac{1}{\pi\sqrt{2}} + \frac{1}{\pi\sqrt{2}}\right) - \left(0 + \frac{1}{\pi}\right)\right] + \left[\left(\frac{1}{\pi\sqrt{2}} + \frac{1}{\pi\sqrt{2}}\right)\right. \\
&\quad \left. - \left(-\frac{1}{\pi\sqrt{2}} - \frac{1}{\pi\sqrt{2}}\right)\right] + \left[\left(0 + \frac{1}{\pi}\right) - \left(-\frac{1}{\pi\sqrt{2}} - \frac{1}{\pi\sqrt{2}}\right)\right] \\
&= \frac{8}{\pi\sqrt{2}} \;=\; 1.8006 \text{ units}^2 .
\end{aligned}
$$

QUESTION 2. Suppose that energy is produced at a rate of

$$E(t) = |300t - 37t^2 + t^3|$$

where E is measured in joules per hour and t is measured in hours. Find the average energy generated between $t = 0$ and $t = 25$. (Hint: The rate is zero at times $t = 0$, 12 and 25.)

[2] Once again, it's a good idea to draw a sketch. In this case it's a cubic, with positive leading term and the hint has kindly given us the places where it crosses the x-axis, so it's quite easy to draw. For the absolute value part, we turn the negative piece upside down, so it looks like this:



Since we have an absolute value, we need to split it into two pieces. The split occurs at $x = 12$, so the first part is the regular (positive) function and the second is the negative of the (negative) function. Thus, the average is

$$\begin{aligned}
\text{Average} &= \frac{1}{25}\left[\int_0^{12} \left(300t - 37t^2 + t^3\right) dt + \int_{12}^{25} \left(37t^2 - 300t - t^3\right) dt\right] \\
&= \frac{1}{25}\left[150t^2 - \frac{37t^3}{3} + \frac{t^4}{4}\right]_0^{12} + \frac{1}{25}\left[\frac{37t^3}{3} - 150t^2 - \frac{t^4}{4}\right]_{12}^{25} \\
&= \frac{1}{25}[5472 - 0] + \frac{1}{25}[-1302.0833 - (-5472)] \\
&= \frac{9641.9176}{25} \\
&= 385.6767 \text{ joules/s}\,.
\end{aligned}$$

QUESTION 3. Consider a skinny snake that is 3 metres long, with a density described by

$$\rho(x) = 3 \times 10^{-5}x - 11 \times 10^{-8}x^2 + 10^{-3}$$

where ρ is measured in kilograms per centimetre and x is measured in centimetres from the tip of the tail.

a) Find the minimum density of the snake.

b) Find the maximum density of the snake.

c) Where does the maximum occur?

d) Where does the minimum occur?

e) Find the total mass of the snake.

f) Find the average density of the snake.

g) How does the average density compare with the minimum and maximum?

h) Graph the density and average.

The derivative is

$$\rho'(x) = 3 \times 10^{-5} - 22 \times 10^{-8}x \qquad \text{(using the chain rule)}\,.$$

The turning point occurs when $\rho'(x) = 0$. Thus $x = \frac{3\times10^{-5}}{22\times10^{-8}} = 136.3636$.

Maxima or minima can only occur at either turning points or on the boundary. We have $\rho(0) = 0.001$, $\rho(136.3636) = 0.003$ and $\rho(300) = 0.0001$.

(a) Thus, the maximum is 0.003. **(0.5 points)**

(b) The minimum is 0.0001.

(c) The maximum occurs at $x = 136.3636$.

(d) The minimum occurs at $x = 300$. **(0.5 points)**

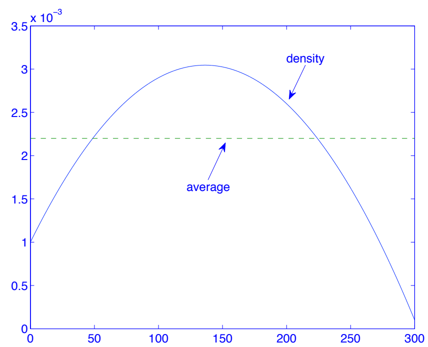
(e) The total mass is

$$\begin{aligned}\text{Total mass} &= \int_0^{300} \rho(x)dx \qquad \text{(remember 3m=300cm)}\\ &= \int_0^{300} \left[3\times 10^{-5}x - 11\times 10^{-8}x^2 + 10^{-3}\right] dx\\ &= \left[\frac{3\times 10^{-5}x^2}{2} - \frac{11\times 10^{-8}x^3}{3} + 10^{-3}x\right]_0^{300}\\ &= 0.66 \text{ kg}\,.\end{aligned}$$

(f) The average mass is just $\frac{0.66}{300} = 0.0022$ kg. **(0.5 points)**

(g) The average is the midpoint between the minimum and maximum densities.

(h) The graph is



(0.5 points)

Question 4. A less-skinny 3-metre-long snake's body is given by rotating the function

$$y = \frac{1}{7}e^{-0.003x}$$

around the x-axis (the units of x and y are in centimetres). What is the volume of this snake?

The volume obtained by rotating $f(x) = \frac{e^{-0.003x}}{7}$ around the x-axis, between 0 and 300 is given by the formula

$$\begin{aligned}V &= \pi \int_0^{300} [f(x)]^2\, dx\\ &= \pi \int_0^{300} \frac{e^{-0.006x}}{49} dx\\ &= \frac{\pi}{49}\int_0^{300} e^{-0.006x}dx\\ &= \frac{\pi}{49}\left[\frac{e^{-0.006x}}{-0.006}\right]_0^{300}\\ &= \frac{\pi}{49}\left[\frac{e^{-1.8}}{-0.006} + \frac{1}{0.006}\right]\\ &= 8.9194\text{cm}^3\end{aligned}$$

QUESTION 5. Find the volume obtained by rotating $f(x) = \tan\left(\frac{x}{4}\right)$ around the x-axis between $-\pi$ and π. (Hint: Find the derivative of $g(\theta) = \tan(a\theta)$, where a is a constant.)

[2] The volume obtained by rotating $f(x) = \tan\left(\frac{x}{4}\right)$ around the x-axis, between $-\pi$ and π is given by the formula

$$\begin{aligned} V &= \pi \int_{-\pi}^{\pi} [f(x)]^2\, dx \\ &= \pi \int_{-\pi}^{\pi} \tan^2\left(\frac{x}{4}\right) dx \\ &= \pi \int_{-\pi}^{\pi} \frac{\sin^2\left(\frac{x}{4}\right)}{\cos^2\left(\frac{x}{4}\right)} dx \\ &= \pi \int_{-\pi}^{\pi} \frac{1-\cos^2\left(\frac{x}{4}\right)}{\cos^2\left(\frac{x}{4}\right)} dx \\ &= \pi \int_{-\pi}^{\pi} \left(\sec^2\left(\frac{x}{4}\right) - 1\right) dx \end{aligned}$$

As noted in the hint,

$$\begin{aligned} \frac{d}{d\theta}\tan(a\theta) &= \frac{d}{d\theta}\left(\frac{\sin(a\theta)}{\cos(a\theta)}\right) \\ &= \frac{a\cos(a\theta)\cos(a\theta) + a\sin(a\theta)\sin(a\theta)}{\cos^2(a\theta)} && \text{(using the quotient rule)} \\ &= \frac{a}{\cos^2(a\theta)} && \text{(since } \cos^2(a\theta)+\sin^2(a\theta) = 1) \\ &= a\sec^2(a\theta) \end{aligned}$$

Hence

$$\int \sec^2(a\theta) = \frac{1}{a}\tan\theta + C$$

We thus have

$$\begin{aligned} V &= \pi \int_{-\pi}^{\pi} \left(\sec^2\left(\frac{x}{4}\right) - 1\right) dx \\ &= \pi \left[4\tan\left(\frac{x}{4}\right) - x\right]_{\pi}^{\pi} \\ &= \pi \left[4\tan\left(\frac{\pi}{4}\right) - \pi\right] - \left[4\tan\left(-\frac{\pi}{4}\right) + \pi\right] \\ &= \pi(8-2\pi) \\ &= 5.3935 \text{ units}^3. \end{aligned}$$