MAT 1332, Fall 2017, Assignment 1

Due Wednesday September 27 in the math department dropboxes by 10:00pm.

Late assignments will not be accepted; nor will unstapled assignments.

Professors in the math department will not lend you a stapler; do not ask for one.

Please print double sided to save paper.

Name (Prime student) ______________________________ Student Number ________________________

Student Name ______________________________________ Student Number ________________________

Student Name ______________________________________ Student Number ________________________

By signing below, we declare that this work is our own, that we have not copied from any other individual or other source and that all students contributed equally.

Signatures ____

QUESTION 1. Calculate

(a) $\displaystyle\int \frac{1}{3-14t}dt$

(2 points) Use the substitution $u = 3 - 14t$. Then $\frac{du}{dt} = -14$, so $dt = \frac{du}{-14}$. Thus

$$\int \frac{1}{3-14t}dt = -\frac{1}{14}\int \frac{1}{u}du = -\frac{1}{14}\ln|u| + C = -\frac{1}{14}\ln|3-14t| + C$$

(0.5 marks for the substitution, 0.5 for the answer, 0.5 for the absolute value signs and 0.5 for the $+C$)

(b) $\displaystyle\int_{-3}^{3}(y^7 - 2y^9)dy$

$$\int_{-3}^{3}(y^7 - 2y^9)dy = \left[\frac{y^8}{8} - \frac{y^{10}}{5}\right]_{-3}^{3} = \left[\frac{3^8}{8} - \frac{3^{10}}{10}\right] - \left[\frac{(-3)^8}{8} - \frac{(-3)^{10}}{10}\right] = 0\,.$$

QUESTION 2. Calculate the following integrals. Leave your answers in exact form (not a decimal approximation).

(a) $\displaystyle\int_{-\pi}^{\pi}[x^2 - 30\cos(x)]dx$

$$\int_{-\pi}^{\pi}[x^2 - 30\cos(x)]dx = \left[\frac{x^3}{3} - 30\sin x\right]_{-\pi}^{\pi} = \left[\frac{\pi^3}{3} - 0\right] - \left[\frac{(-\pi)^3}{3} - 0\right] = \frac{2\pi^3}{3}\,.$$

(b) $3\int_2^5 \sin(3\pi(x-5)dx$

(2 points) *First approach* First find the indefinite integral by using the substitution $u = 3\pi(x-5)$. Then $\frac{du}{dx} = 3\pi$, so $dx = \frac{du}{3\pi}$. Hence

$$3\int \sin(3\pi(x-5)dx = 3\int \sin u \frac{du}{3\pi} = -\frac{1}{\pi}\cos u + C = -\frac{1}{\pi}\cos(3\pi(x-5)) + C\,.$$

Then evaluate

$$3\int_2^5 \sin(3\pi(x-5)dx = -\frac{1}{\pi}\cos(3\pi(x-5))\Big|_2^5 = -\frac{1}{\pi}\left[\cos(0) - \cos(-9\pi)\right] = -\frac{1}{\pi}[1+1] = -\frac{2}{\pi}\,.$$

Second approach Transform the limits of integration first. When $x = 2$, $u = 3\pi(2-5) = -9\pi$. When $x = 5$, $u = 3\pi(5-5) = 0$. Then the integral after substitution becomes

$$3\int_2^5 \sin(3\pi(x-5)dx = 3\int_{-9\pi}^0 \sin u \frac{du}{3\pi} = -\frac{1}{\pi}\left[\cos(0) - \cos(-9\pi)\right] = -\frac{2}{\pi}\,.$$

(2 marks for the answer; no part marks. Zero marks if the decimal approximation is used.)

Question 3. Calculate

$$\int_0^1 \frac{e^{\arctan x}}{1+x^2}dx\,.$$

Use the substitution $u = \arctan x$. Then $\frac{du}{dx} = \frac{1}{1+x^2}$ so $dx = (1+x^2)du$. Hence

$$\int_0^1 \frac{e^{\arctan x}}{1+x^2}dx = \int_{x=0}^{x=1} \frac{e^u}{1+x^2}(1+x^2)du = \int_{x=0}^{x=1} e^u du = e^u\Big|_{x=0}^{x=1} = e^{\arctan x}\Big|_0^1 = e^{\pi/4} - 1 = 0.284025\,.$$

Question 4. Calculate

$$\int_0^\pi x\sin(3x)dx\,.$$

(2 points) Using integration by parts, we have

$$u = x \qquad v' = \sin 3x$$

$$u' = 1 \qquad v = -\frac{\cos 3x}{3}$$

Thus

$$\begin{aligned}\int_0^\pi x\sin(3x)dx &= -\frac{1}{3}x\cos 3x\Big|_0^\pi + \frac{1}{3}\int_0^\pi \cos 3x dx \\ &= \left[-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x\right]_0^\pi \\ &= \left[-\frac{1}{3}\pi\cos 3\pi + \frac{1}{9}\sin 3\pi\right] - \left[-\frac{1}{3}(0) - \frac{1}{9}\sin 0\right] \\ &= \frac{\pi}{3}\end{aligned}$$

(2 marks for the answer; no part marks.)

Question 5. Zombies have invaded campus! Initially, there are 5 zombies. They recruit more of the undead to their ghoulish ranks at rate

$$\frac{dz}{dt} = 10te^{-0.08t},$$

where t is the time in days and z are the number of zombies.

(a) How many zombies are recruited in the first week? 170 **(1 point; no part marks)**

(b) How many zombies are recruited during the third week? 300 **(1 point; no part marks)**

(c) After 50 days, how many zombies are there in total? 1424 **(1 point; no part marks)**

(d) Will zombies eventually infect everyone on campus? If not, how many will be infected eventually? No. 1567. **(2 points, 1 for each answer; no part marks)**

(Please include your work below or on extra pages, but be sure to write your final answers in the spaces above.)

To answer the questions, we need to solve the zombie equation. Since we'll need to do this for a variety of scenarios, it's best to find the indefinite integral first. We'll need to use integration by parts, so we have

$$u = 10t \qquad v' = e^{-0.08t}$$

$$u' = 10 \qquad v = -\frac{e^{-0.08t}}{0.08}$$

Thus

$$\begin{aligned} z &= -\frac{10te^{-0.08t}}{0.08} + \frac{10}{0.08}\int e^{-0.08t}dt \\ &= -\frac{10te^{-0.08t}}{0.08} - \frac{10}{0.08^2}e^{-0.08t} + C \end{aligned}$$

Don't forget the $+C$, it's absolutely crucial. Forget the $+C$ and you'll be devoured by a zombie.

What happens now? We almost have the solution, but not quite (we don't know what C is). And we haven't used the initial condition $z(0) = 5$. Let's put that in and see what happens:

$$z(0) = -0 - \frac{10}{0.08^2}e^0 + C = -\frac{10}{0.08^2} + C = 5\,.$$

Thus,

$$C = 5 + \frac{10}{0.08^2}\,.$$

and hence

$$z(t) = -\frac{10te^{-0.08t}}{0.08} - \frac{10}{0.08^2}e^{-0.08t} + 5 + \frac{10}{0.08^2}\,.$$

This is the solution because it tells us how many zombies exist at any given time. There's only z and t to be determined; everything else is known. We're now in a position to answer our sub-questions.

a) During the first week, there will be $z(7) - z(0)$ zombies recruited.

$$z(7) - z(0) = -\frac{10(7)e^{-0.08(7)}}{0.08} - \frac{10}{0.08^2}e^{-0.08(7)} + 5 + \frac{10}{0.08^2} - 5 = 170.17$$

Thus, there are 170 zombies recruited during the first week. They may be undead, but the zombies are very efficient.

(It's not 175, since the original 5 weren't recruited. That's why we have to subtract $z(0)$.)

b) **(1 point)** During the third week there are $z(21)$ zombies in total, but $z(14)$ of them already existed at the end of the second week. Thus, during the third week, $z(21)-z(14)$ zombies are recruited.

$$\begin{aligned} z(21) - z(14) &= \left[-\frac{10(21)e^{-0.08(21)}}{0.08} - \frac{10}{0.08^2}e^{-0.08(21)} + 5 + \frac{10}{0.08^2}\right] \\ &\quad - \left[-\frac{10(14)e^{-0.08(14)}}{0.08} - \frac{10}{0.08^2}e^{-0.08(14)} + 5 + \frac{10}{0.08^2}\right] \\ &= 300.3608\,. \end{aligned}$$

Thus, there are 300 zombies recruited in the third week. This is a lot; it's getting dangerous on campus!

(And you can see the fundamental theorem of calculus at work here. Now which is scarier: math or zombies?)

c) After 50 days, the total number of zombies is

$$z(50) = -\frac{10(50)e^{-0.08(50)}}{0.08} - \frac{10}{0.08^2}e^{-0.08(50)} + 5 + \frac{10}{0.08^2} = 1424.4091\,.$$

Fifty days into the semester, if you visit campus, you'll see a lot of pale, shambling figures, their eyes glazed over and making moaning sounds. You'll also see 1424 zombies.

d) **(2 points)** What do we mean by "eventually"? We sort of mean "after a very long time" but that's not formal enough for calculus class. What we really mean is "after an infinite amount of time". Thus, we need to take the limit as time goes to infinity.

(Remember, infinity is a useful abstraction and only makes sense in the context of a limit.) Thus,

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \left[-\frac{10te^{-0.08t}}{0.08} - \frac{10}{0.08^2}e^{-0.08t} + 5 + \frac{10}{0.08^2} \right] .$$

The last three terms aren't a problem, but the first one might be, because

$$\lim_{t\to\infty} -\frac{10te^{-0.08t}}{0.08} = \infty \times 0$$

which we *cannot* evaluate as is. It's an indeterminant form and there's only one way to deal with those: L'Hôpital's rule. Which means we need to have it in a fractional form. Thus

$$\lim_{t\to\infty} -\frac{10te^{-0.08t}}{0.08} = \lim_{t\to\infty} -\frac{10t}{0.08e^{0.08t}} = \frac{\infty}{\infty} .$$

Hence, we can apply L'Hôpital's rule:

$$\lim_{t\to\infty} -\frac{10t}{0.08e^{0.08t}} = \lim_{t\to\infty} -\frac{10}{0.08^2e^{0.08t}} = 0 .$$

We thus have

$$\begin{aligned} \lim_{t\to\infty} z(t) &= \lim_{t\to\infty} \left[-\frac{10te^{-0.08t}}{0.08} - \frac{10}{0.08^2}e^{-0.08t} + 5 + \frac{10}{0.08^2} \right] \\ &= 0 - 0 + 5 + \frac{10}{0.08^2} = 1567.5 \end{aligned}$$

So now we know: the zombies will wreak some devastation, infecting 1567 people, but they won't take over the whole campus.

[Note that it's not 1568, because you never round up when considering individuals. You can't have half a zombie. That would be highly unrealistic!]

Calculus can help you avoid the impending zombie plague. Who said math was useless?