MATH3705 D – Test 2: Friday, Feb. 14, 14:35–15:25

Name and Student Number:

Total points: 15. No partial marks for Questions 1-4.

Closed book! Non-programmer calculators are allowed!

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1. (1 point) Find the general solution of $x^2y'' - 4xy' + 6y = 0$ for $x \neq 0$.

   (a) $c_1|x|^{-2} + c_2|x|^3$ (b) $c_1|x|^{-2} + c_2|x|^{-3}$ (c) $c_1|x|^{-1} + c_2|x|^{-5}$
   (d) $|x|^2[c_1\cos(\sqrt{2}\ln|x|) + c_2\sin(\sqrt{2}\ln|x|)]$ (e) $c_1|x|^2 + c_2|x|^3$

   Solution: (e)
   The indicial equation is
   $$r^2 - 5r + 6 = 0, \Rightarrow r = 2, 3.$$
   The general solution is
   $$y(x) = c_1|x|^2 + c_2|x|^3.$$

2. (1 point) Find the general solution of $x^2y'' + 5xy' + 4y = 0$ for $x \neq 0$.

   (a) $c_1|x|^2 + c_2|x|^{-2}\ln|x|$ (b) $c_1|x|^{-2} + c_2|x|^2\ln|x|$ (c) $c_1|x|^2 + c_2|x|^2\ln|x|$
   (d) $c_1|x|^2 + c_2|x|^{-2}$ (e) $c_1|x|^{-2} + c_2|x|^{-2}\ln|x|$

   Solution: (e)
   The indicial equation is
   $$r^2 + 4r + 4 = 0, \Rightarrow r = -2.$$
   The general solution is
   $$y(x) = c_1|x|^{-2} + c_2|x|^{-2}\ln|x|.$$

3. (1 point) Which of the following is a solution of $x^2y'' + 5xy' + 8y = 0$ for $x > 0$?

   (a) x^2 (b) $x^2\sin(4\ln x)$ (c) x^{-2} (d) $x^{-2}\sin(\ln x)$ (e) $x^{-2}\sin(2\ln x)$

Solution: (e)

The indicial equation is

$$r^2 + 4r + 8 = 0, \Rightarrow r = -2 \pm 2i.$$

The general solution is

$$y(x) = x^{-2}[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)].$$

4. (1 point) Which of the following is a solution of $x^2 y''(x) + xy'(x) + (x^2 - 2.25)y(x) = 0$, for $x > 0$?

(a) $J_{-1.5}(x)$ (b) $J_1(1.5x)$ (c) $Y_{2.25}(x)$ (d) $Y_1(2.25x)$ (e) $Y_1(1.5x)$

Solution: (a). Note that $\lambda^2 = 1$ and $\nu^2 = 2.25$, $\Rightarrow \lambda = 1$ and $\nu = 1.5$. Hence

$$y_1(x) = J_{1.5}(x), \quad y_2(x) = J_{-1.5}(x).$$

5. (5 points) Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the series solution of $(x^2 + 1)y'' + xy' + y = 0$. Find the relation between a_{n+2} and a_n.

Solution: From $y = \sum_{n=0}^{\infty} a_n x^n$ we have

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1},$$

and

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Substitute all of them into the DE,

$$(x^2 + 1) \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0. \Rightarrow$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0. \Rightarrow$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0. \Rightarrow$$

$$\sum_{n=0}^{\infty}[n(n-1)a_n + na_n + a_n + (n+2)(n+1)a_{n+2}]x^n = 0. \Rightarrow$$

$$n(n-1)a_n + na_n + a_n + (n+2)(n+1)a_{n+2} = 0, \Rightarrow$$

$$a_{n+2} = -\frac{n^2+1}{(n+1)(n+2)}a_n.$$

6. (6 points) Let $y = \sum_{n=0}^{\infty} c_n(r)x^{n+r}$, $c_0(r) = 1$ be the solution of the following DE:

$$xy'' + (x-0.5)y' - 0.5y = 0$$

for $x > 0$ near $x_0 = 0$. The recursive relation is: $c_{n+1}(r) = \frac{-1}{n+r+1}c_n(r), n \geq 0$.
(i) (2 points) Write down the indicial equation and solve it to determine r_1 and r_2, $r_1 \geq r_2$.

Solution: We rewrite the DE as

$$y'' + \frac{x-0.5}{x}y' - \frac{0.5}{x}y = 0.$$

Then

$$xp(x) = -0.5 + x, \quad x^2q(x) = -0.5x.$$

Thus $p_0 = -0.5, q_0 = 0$. The indicial equation is:

$$r^2 + (p_0 - 1)r + q_0 = 0. \Rightarrow r^2 - 1.5r = 0. \Rightarrow r_1 = 1.5, r_2 = 0.$$

Note that $r_1 - r_2 = 1.5$, so we have Case (i).
(ii) (4 points = 2+2) Solve $c_n(r_1)$ and $c_n(r_2)$.

Solution: Take $r = r_1 = 1.5$, by the recursive equation,

$$c_{n+1}(1.5) = \frac{-1}{n+2.5}c_n(r), n \geq 0.$$

$$c_1(1.5) = \frac{-1}{2.5}c_0(1.5) = \frac{-1}{2.5},$$

$$c_2(1.5) = \frac{-1}{3.5}c_1(1.5) = \frac{(-1)^2}{2.5(3.5)}, ...,$$

$$c_n(1.5) = \frac{(-1)^n}{2.5(3.5)...(n+1.5)}, \quad or \quad \frac{(-2)^n}{5(7)...(2n+3)}, \quad n \geq 1.$$

Take $r = r_2 = 0$, by the recursive equation,

$$c_{n+1}(0) = \frac{-1}{n+1} c_n(r), n \geq 0.$$

$$c_1(0) = \frac{-1}{1} c_0(0) = \frac{-1}{1},$$

$$c_2(0) = \frac{-1}{2} c_1(0) = \frac{(-1)^2}{1(2)}, ...,$$

$$c_n(0) = \frac{(-1)^n}{1(2)...(n)} = \frac{(-1)^n}{n!}, n \geq 1.$$