Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Midterm 1 for MAT 1339B (Winter 2018)
Introduction to Calculus and Vectors

Duration: 80 minutes Professor: Rachid Bentoumi

NAME: ______________ STUDENT NUMBER: ______________

You must sign below Version 2

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- This is a closed book examination.
- Only Faculty standard calculators are permitted.
- There are 6 questions.
- The exam is out of 30 points.

Question 1. Calculate the following limits.

[2] 1. $\lim\limits_{x \to 4} \dfrac{x^2 - 6x + 8}{x - 4}$ $\left(\frac{0}{0}\right)$ I.F

$$\lim_{x \to 4} \frac{x^2 - 6x + 8}{x - 4} = \lim_{x \to 4} \frac{\cancel{(x-4)}(x-2)}{\cancel{(x-4)}}$$

$$= \lim_{x \to 4} (x-2) = 4 - 2 = 2$$

[2] 2. $\lim\limits_{x \to \infty} \dfrac{\cos(x)}{x^8 + 3}$

$$-1 \leq \cos(x) \leq 1$$

$$\frac{-1}{x^8+3} \leq \frac{\cos(x)}{x^8+3} \leq \frac{1}{x^8+3}$$

$$\Rightarrow \lim_{x \to \infty} \frac{-1}{x^8+3} \leq \lim_{x \to \infty} \frac{\cos(x)}{x^8+3} \leq \lim_{x \to \infty} \frac{1}{x^8+3}$$

$$\to 0$$

By Sandwich Theorem, we have $\lim\limits_{x \to \infty} \dfrac{\cos(x)}{x^8+3} = 0$

Question 2. Consider the following function

$$f(x) = \frac{x-4}{3-x^2}$$

[1] 1. Find the domain of f.

$$D_f = \{x \in \mathbb{R} \mid 3 - x^2 \neq 0\} = \{x \in \mathbb{R} \mid (\sqrt{3}-x)(\sqrt{3}+x) \neq 0\}$$

$$= \{x \in \mathbb{R} \mid x \neq \sqrt{3} \text{ and } x \neq -\sqrt{3}\}$$

$$= \mathbb{R} \setminus \{-\sqrt{3}, \sqrt{3}\}$$

[2] 2. Calculate the rate of change of f on $[-1, 1]$

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{\frac{1-4}{3-1} - \frac{-1-4}{3-(-1)^2}}{1+1}$$

$$= \frac{\frac{-3}{2} - \frac{-5}{2}}{2} = \frac{\frac{-3+5}{2}}{2} = \frac{1}{2} = \frac{1}{2}$$

Question 3. Let g be a function defined as: $g(x) = \sqrt{3-4x}$.

[3] 1. Use the **definition of the derivative** to find the derivative of g

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3-4(x+h)} - \sqrt{3-4x}}{h} \quad \left(\frac{0}{0}\right) \text{ I.F}$$

$$= \lim_{h\to 0} \frac{\sqrt{3-4x-4h} - \sqrt{3-4x}}{h} \times \frac{\sqrt{3-4x-4h} + \sqrt{3-4x}}{\sqrt{3-4x-4h} + \sqrt{3-4x}}$$

$$= \lim_{h\to 0} \frac{(\cancel{3}-\cancel{4x}-4h) - (\cancel{3}-\cancel{4x})}{h\left(\sqrt{3-4x-4h} + \sqrt{3-4x}\right)} = \lim_{h\to 0} \frac{-4\cancel{h}}{\cancel{h}\left(\sqrt{3-4x-4h}+\sqrt{3-4x}\right)}$$

$$= \frac{-4}{\sqrt{3-4x} + \sqrt{3-4x}} = \frac{-4}{2\sqrt{3-4x}} = \frac{-2}{\sqrt{3-4x}}$$

[2] 2. Find the tangent line to g at $x = 0$.

$y = mx + b$, at $x=0$ $\quad g(0) = \sqrt{3-4(0)} = \sqrt{3}$

$$m = g'(0) = \frac{-2}{\sqrt{3-4(0)}} = \frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$$

$$y = -\frac{2}{\sqrt{3}}x + b$$

at $x=0$, $y = g(0) = \sqrt{3}$

so, $\sqrt{3} = \frac{-2}{\sqrt{3}}(0) + b$

$\Rightarrow b = \sqrt{3}$

$$\Rightarrow y = -\frac{2}{\sqrt{3}}x + \sqrt{3} = -\frac{2\sqrt{3}}{3}x + \sqrt{3}.$$

Question 4. Consider the following function

$$f(x) = \begin{cases} 9 - x^2, & \text{if } x \leq 3 \\ \\ kx - 3, & \text{if } x > 3 \end{cases}$$

[3] 1. Determine the value of k that makes the function f continuous on $\mathbb{R}$.

$* \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 9 - x^2 = 9 - (3)^2 = 0$

$* \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} Kx - 3 = 3K - 3$

$* f(3) = 9 - (3)^2 = 0$

f is continuous $\forall x \in \mathbb{R}$. In particular it is continuous at $x = 3$. So that

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$

$\Rightarrow \quad 0 = 3K - 3 = 0 \Rightarrow 3K = 3 \Rightarrow K = 1$

[1] 2. Suppose that $k = -3$. Can we conclude that the derivative of f exists at $x = 3$? Justify your answer.

Method 1 From 1) f is continuous at $x = 3$ if $K = 1$.

Now, if $K = -3$ then f is not continuous at $x = 3$. This means that f is not differentiable at $x = 3$. ($f'(3)$ does not exist)

Method 2 if $K = -3$ then

$f'(x) = \begin{cases} -2x & \text{if } x \leq 3 \\ -3 & \text{if } x > 3 \end{cases} \Rightarrow f'(3) = \begin{cases} -6, & x \leq 3 \\ -3, & x > 3 \end{cases}$

two different slops $\Rightarrow f'(3)$ does not exist.

Question 5. Use the **derivative rules** to calculate the derivatives of the functions below.

[3] 1. $y = \sqrt{7x^2 - 5x - 28}$.

$$y' = \left(\left(7x^2 - 5x - 28\right)^{1/2}\right)'$$
$$= \frac{1}{2}\left(7x^2 - 5x - 28\right)'\left(7x^2 - 5x - 28\right)^{1/2 - 1}$$
$$= \frac{1}{2}\left(14x - 5\right)\left(7x^2 - 5x - 28\right)^{-1/2}$$
$$= \frac{14x - 5}{2\sqrt{7x^2 - 5x - 28}}$$

[3] 2. $y = x^5(x^3 - 7)^4$.

$$y' = (x^5)'(x^3 - 7)^4 + x^5\left[(x^3 - 7)^4\right]'$$
$$= 5x^4(x^3 - 7)^4 + x^5(4)(x^3 - 7)'(x^3 - 7)^{4-1}$$
$$= 5x^4(x^3 - 7)^4 + 4x^5(3x^2)(x^3 - 7)^3$$
$$= 5x^4(x^3 - 7)^4 + 12x^7(x^3 - 7)^3$$

[3] 3. $y = \dfrac{x - 3}{x^2 + 5}$.

$$y' = \frac{(x-3)'(x^2+5) - (x-3)(x^2+5)'}{(x^2+5)^2}$$
$$= \frac{(1)(x^2+5) - (x-3)(2x)}{(x^2+5)^2}$$

$$= \frac{(x^2+5) - (x-3)(2x)}{(x^2+5)^2}$$

Question 6. A manufacturer estimates that his weekly revenue $R(x)$ can be expressed by

$$R(x) = 1008x - 12x^2 - 8x^3$$

where x is the number of units sold, $R(x)$ is expressed in dollars. Under certain constraints, one cannot produce more than 10 units per week.

[5] Use the **Second Derivative Test** to determine the quantity of units that gives a maximum revenue and evaluate this maximum revenue.

Solution :

See Version 1

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