MAT1339 B - Winter 2018

Lecture - 6

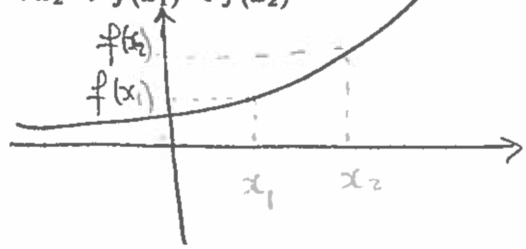
1 Applications of the derivative

1.1 Increasing and Decreasing Functions

One of our goals is to be able to solve max/min problems, especially economics related examples. We start with the following definitions:

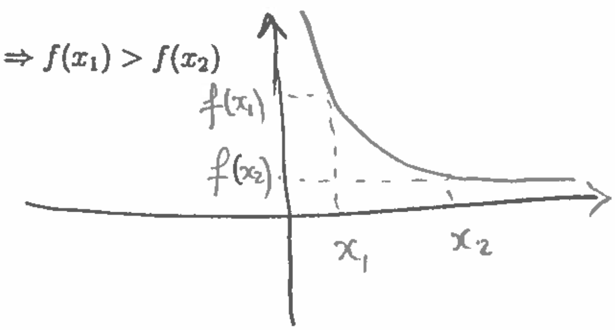
Definition 1.1 *A function f is called* **increasing** *on an interval (a,b) if for any $x_1, x_2 \in (a,b)$, we have that*

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$



A function f is called **decreasing** *on an interval (a,b) if for any $x_1, x_2 \in (a,b)$, we have that*

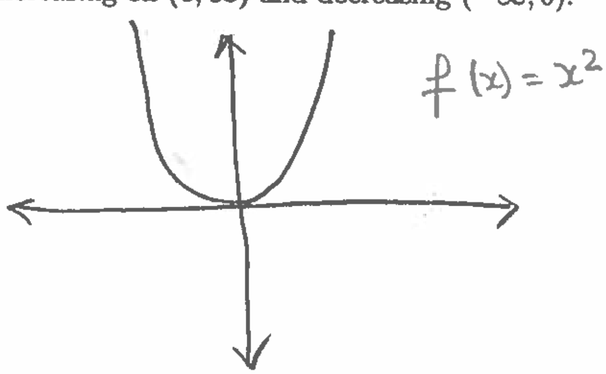
$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$



Note that some books call these **strictly increasing** and **strictly decreasing** respectively.

Examples:

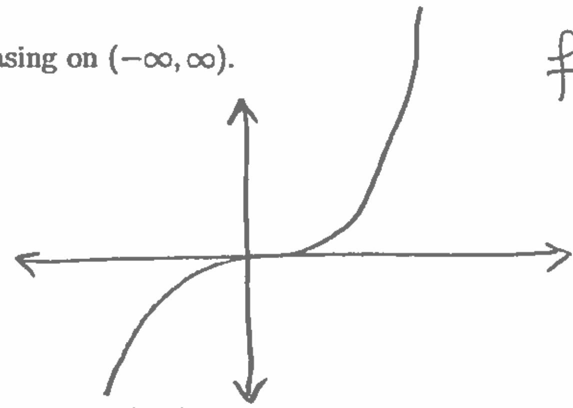
1. $f(x) = x^2$. $f(x)$ is increasing on $(0, \infty)$ and decreasing $(-\infty, 0)$.



$f(x)$ is $\nearrow$ on $(0,\infty)$

$f(x)$ is $\searrow$ on $(-\infty,0)$

2. $f(x) = x^3$. $f(x)$ is increasing on $(-\infty, \infty)$.



$f(x) = x^3$ is $\nearrow$ on $(-\infty, \infty)$

As you may have guessed, we can use the derivative to test for increasing/decreasing. Let f be differentiable on the interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b).
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).
3. If $f'(x) = 0$ for all x in (a, b), then f must be constant on (a, b).

Examples:

1. Let $f(x) = \frac{x^3}{4} - 3x$. Find the intervals on which $f(x)$ is increasing or decreasing.

$D_f = (-\infty, \infty)$

$$f'(x) = \frac{3}{4}x^2 - 3 = \frac{3}{4}(x^2 - 4) = \frac{3}{4}(x-2)(x+2)$$

$f'(x) = 0 \Rightarrow x - 2 = 0$ or $x + 2 = 0$

$\Rightarrow x = 2$ or $x = -2$

Method 1

x	$-\infty$		-2		2		∞
$x-2$		$-$	$-$		0	$+$	
$x+2$		$-$	0	$+$		$+$	
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$		$\nearrow$		$\searrow$		$\nearrow$	

f is increasing on $(-\infty, -2) \cup (2, \infty)$

f is decreasing on $(-2, 2)$

Method 2

	$f'(-3)>0$	-2	$f(0)<0$	2	$f'(3)>$
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$

2. Let $g(x) = x + \frac{32}{x^2}$. Find the intervals on which $g(x)$ is increasing and decreasing.

$D_g = (-\infty, 0) \cup (0, \infty)$. $g'(x) = (x + 32 \cdot x^{-2})' = 1 + 32(-2)x^{-3}$
$= 1 - \frac{64}{x^3} = \frac{x^3 - 64}{x^3}$

$g'(x) = 0 \Rightarrow x^3 - 64 = 0 \Rightarrow x = (64)^{1/3} = 4$

$-\infty$	-1	0	1	4	5 $+\infty$
	$g'(-1) = 65 > 0$		$g'(1) = -63 < 0$		$g'(5) \doteq \frac{61}{125} > 0$
$f'(x)$	$+$		$-$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$

f is $\nearrow$ on $(-\infty, 0) \cup (4, \infty)$ and $f \searrow$ on $(0, 4)$

1.2 Critical points (values)

Definition 1.2 *We call $x = c$ a* **critical point** *or* **critical value** *of the function f with the domain D_f, if*

1. $c \in D_f$
2. $f'(c) = 0$ *or* $f'(c)$ *doesn't exist*

<u>Examples:</u>

1. $f(x) = x^2$. ($x = 0$ is a critical point). First, $D_f = (-\infty, \infty)$

   $f'(x) = 2x$

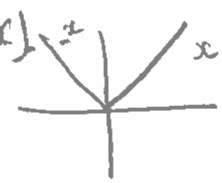
   $f'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0 \in D_f$. So that $x = 0$ is a critical point of f. In addition $f'(x)$ exists $\forall x \in D_f$.

2. $f(x) = |x|$. ($x = 0$ is a critical point because $f'(0)$ does not exist).

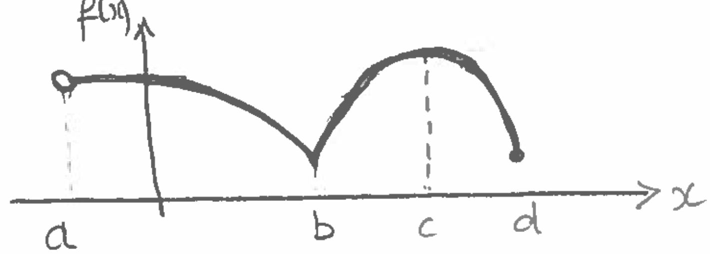
   $D_f = (-\infty, \infty)$. $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \Rightarrow f'(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

   * There is no $x \in D_f$ such that $f'(x) = 0$

   * $f'(0)$ does not exist. In addition, $0 \in D_f$. So that $x = 0$ is the only critical point of $f(x)$.



3. Let $f(x)$ be a function by the following graph



$D_f =]a, d]$

What does the values a, b, c and d represent?

* a is not a critical point since $a \notin D_f$
* b is is a critical point since, $b \in D_f$ and $f'(b)$ does not exist.
* c is a critical point since, $c \in D_f$ and $f'(c) = 0$
* d is a critical point since $d \in D_f$ and $f'(d)$ does not exist

As we can see, it is very easy to determine the critical values of a function using its graphical representation. However, to find these values without drawing the graph of the function, one must take the derivative the function and find the zeros of the derivative or find the points where the derivative does not exist.

4. Find the critical values of the function: $g(x) = \sqrt{x+1}$ $\quad D_g = [-1, \infty)$

$$g'(x) = \left((x+1)^{1/2}\right)' = \frac{1}{2}(x+1)'(x+1)^{1/2-1} = \frac{1}{2}(1)(x+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+1}} > 0$$

* There is no $x \in D_f$ such that $g'(x) = 0$
* $g'(x)$ does not exist at $x = -1 \in D_g \Rightarrow x = -1$ is a critical point of $g(x)$

5. Find the critical values of the function: $g(x) = \frac{1}{x}$ $\quad D_g = (-\infty, 0) \cup (0, \infty)$

$$g'(x) = (x^{-1})' = (-1)(x^{-1-1}) = -x^{-2} = -\frac{1}{x^2}$$

* There is no $x \in D_g$ such that $g'(x) = 0$.
* $g'(x)$ does not exist at $x = 0 \notin D_g$. so that $x = 0$ is not a critical point of $g(x)$. There is no C.P.

6. Find the critical values of the function: $g(x) = x^3 - 2x^2 - 16$ $\quad D_g = (-\infty, \infty)$

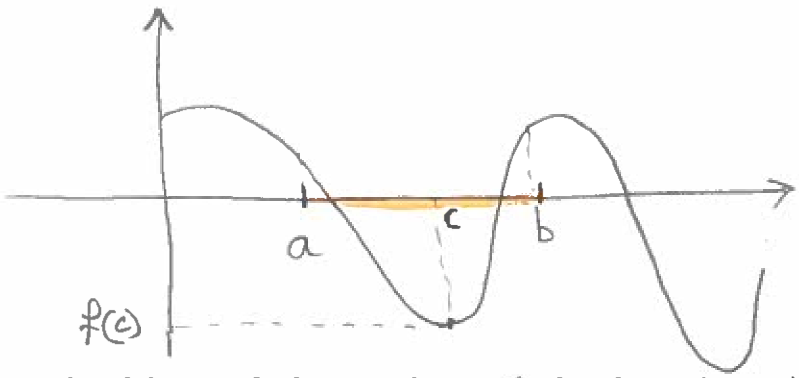
$$g'(x) = 3x^2 - 4x = x(3x - 4) = 0$$

$\Rightarrow x = 0 \in D_g$, $\quad x = \frac{4}{3} \in D_g$. So that, $x = 0$ and $x = \frac{4}{3}$ are two C.P for $g(x)$.

1.3 Extrema and the First Derivative Test

Definition 1.3 *Let f be a function defined at a point c. Then*

1. *We say that f has a* **relative minimum** *(or* **local minimum***) at $x = c$ if there is an interval (a, b) containing c such that $f(c) < f(x)$ for all x in (a, b).*

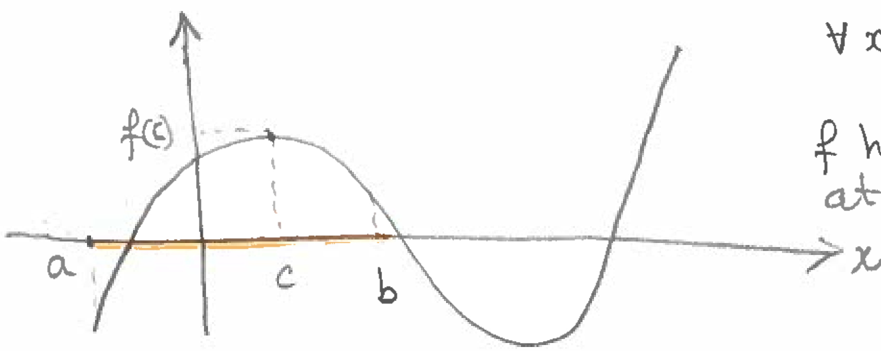


$\forall x \in (a,b)$

$f(c) < f(x)$

f has a relative min at $x=c$

2. *We say that f has a* **relative maximum** *(or* **local maximum***) at $x = c$ if there is an interval (a, b) containing c such that $f(c) > f(x)$ for all x in (a, b).*



$\forall x \in (a,b)$

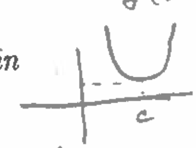
$f(x) < f(c)$

f has a relative max at $x=c$.

Remark Together, relative max and min are called **relative extrema**.

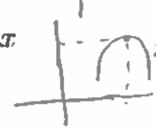
Theorem 1.1 (The First Derivative Test): *Let f be continuous on the interval (a, b) and suppose that c is the only critical point of (a, b). Suppose f is differentiable on (a, b) except possibly at c. Then:*

$f'(c)=0$



1. *If $f'(x)$ is negative to the left of c and positive to the right of c, then f has a local min at c.*

   local min



2. *If $f'(x)$ is positive to the left of c and negative to the right of c, then f has a local max at c.*

   local max

3. *If $f'(x)$ is the same sign to the right and left of c then c is neither a max nor a min.*

In cases like 3 above we call the point c a **saddle point** if $f'(c) = 0$.

$f(x)=x^3$

$f'(x)=3x^2$

$f'(x)=0$

$\Rightarrow x=0$

Here, $f'(0)=0$

Examples: Find the critical points and identify their type for the following functions.

1. $f(x) = 2x^3 + 9x^2 - 108x + 30.$

$D_f = (-\infty, \infty)$

$f'(x) = 6x^2 + 18x - 108$

$= 6(x^2 + 3x - 18) = 6(x+6)(x-3)$

$f'(x) = 0 \Rightarrow (x+6) = 0$ or $x - 3 = 0$

$\Rightarrow x = -6 \in D_f$ or $x = 3 \in D_f$

So that, $x = \boxed{-6}$ and $x = \boxed{3}$ are two c.p for $f(x)$

Test points on $(-\infty, -6) \cup (-6, 3) \cup (3, \infty)$

	-7	$\boxed{-6}$	0	$\boxed{3}$	4	$+\infty$
Test Points	$f'(-7) = 6(-7+6)(-7-3) = 60 > 0$		$f'(0) = -108 < 0$		$f'(4) = 60 > 0$	
$f'(x)$	$+$		$-$		$+$	
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$	

By the first derivative test:

* $f(x)$ has a local max at $x = -6$
* $f(x)$ has a local min at $x = 4$

2. $f(x) = \dfrac{x+1}{x-2}$.

$D_f = \mathbb{R} \setminus \{2\} = (-\infty, 2) \cup (2, \infty)$

$$f'(x) = \left(\frac{x+1}{x-2}\right)' = \frac{(x+1)'(x-2) - (x+1)(x-2)'}{(x-2)^2}$$

$$= \frac{(1)(x-2) - (x+1)(1)}{(x-2)^2}$$

$$= \frac{(x-2) - (x+1)}{(x-2)^2} = \frac{-3}{(x-2)^2} < 0$$

* There is no $x \in D_f$ such that $f'(x) = 0$

* $f'(x)$ does not exist at $x = 2 \notin D_f$.

* We conclude that there is no C.P for $f(x)$

	$-\infty$	2	∞
$f'(x)$	$-$	‖	$-$
$f(x)$	↘	‖	↘