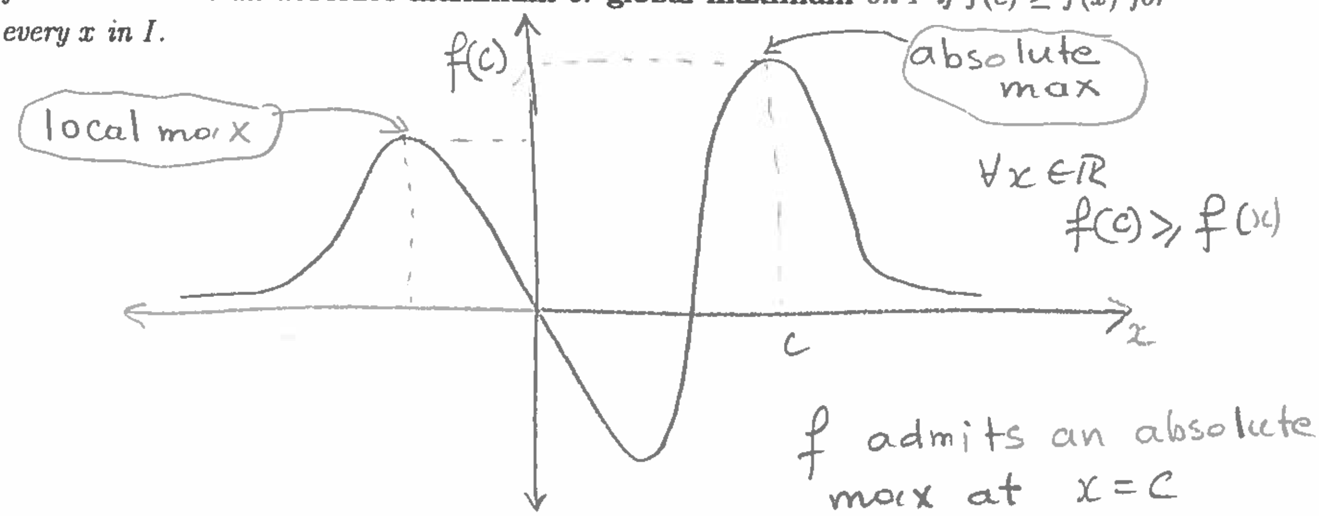
MAT1339 B - Winter 2018

Lecture - 7

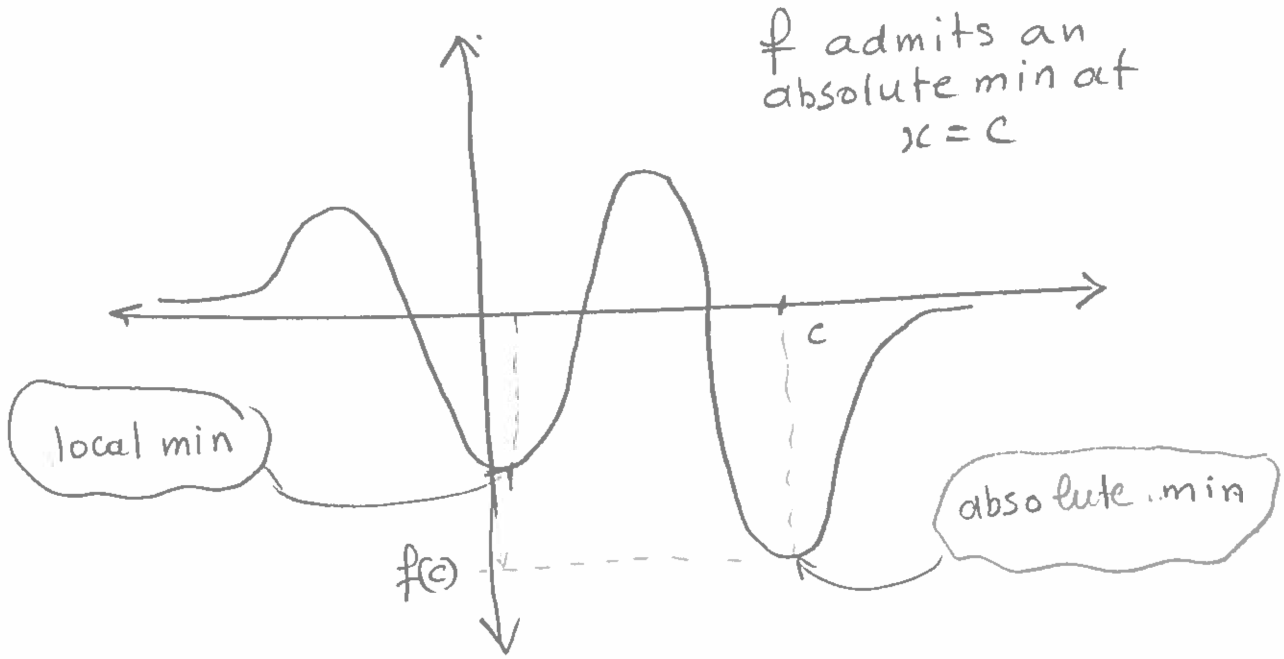
1 Absolute Extrema

In economics applications, we are generally uninterested in local maxes and mins. Instead we want to know when our function is maximal or minimal on its entire domain.

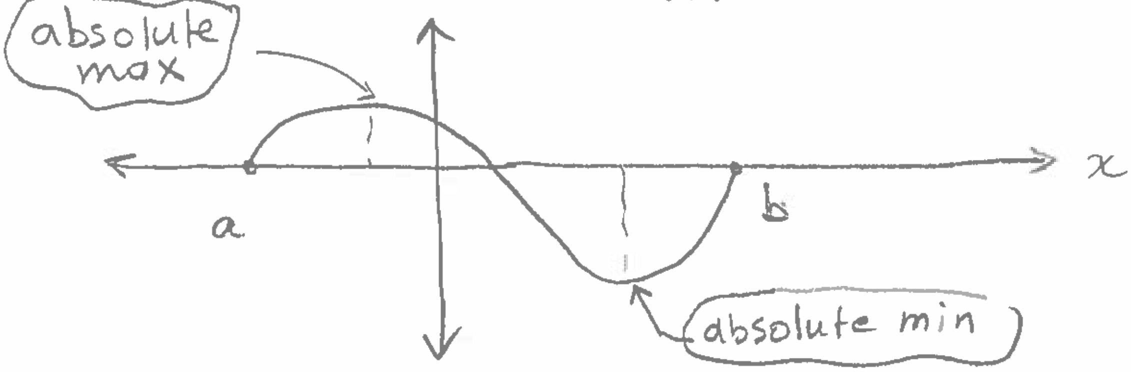
Definition 1.1 *Let f be defined on an interval I (possibly all of $\mathbb{R}$) containing c. Then f is said to have an* **absolute maximum** *or* **global maximum** *on I if $f(c) \geq f(x)$ for every x in I.*



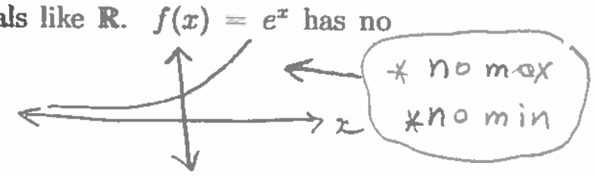
Definition 1.2 *Let f be defined on an interval I (possibly all of $\mathbb{R}$) containing c. Then f is said to have an* **absolute minimum** *or* **global maximum** *on I if $f(c) \leq f(x)$ for every x in I.*



Theorem 1.1 (Extreme Value Theorem): *If f is continuous on a closed interval $[a, b]$, then f has both an absolute max and an absolute min on $[a, b]$.*



Remark: The theorem is certainly false for other intervals like $\mathbb{R}$. $f(x) = e^x$ has no maximum or minimum on $\mathbb{R}$.



To find extrema on a closed interval $[a, b]$ proceed as follows:

1. Find all possible critical points;
2. Evaluate f at each critical point in $[a, b]$;
3. Evaluate f at the endpoints a and b;
4. The least of these values is the absolute min and the greatest is the absolute max.

Examples:

1. Find the absolute max and min of $f(x) = 2x^3 - 3x^2 - 36x + 2$ on the interval $[0, 5]$.

$$f'(x) = (2x^3 - 3x^2 - 36x + 2)' = 6x^2 - 6x - 36$$
$$= 6(x^2 - x - 6) = 6(x+2)(x-3)$$
$$f'(x) = 0 \Rightarrow x + 2 = 0 \text{ or } x - 3 = 0$$
$$\Rightarrow x = -2 \notin [0, 5], \quad x = 3 \in [0, 5]$$

* $x = -2$ is not a critical point

* $x = 5$ is a critical point

* $x = 0$, and $x = 5$ are also critical points since the derivative at endpoints of $[a, b]$ does not exist. ($f'(a)$ and $f'(b)$)

x	$f(x)$		
0	$f(0) = 2$	← abs max	$(0, f(0)) = (0, 2)$
3	$f(3) = -79$		
5	$f(5) = -3$	← abs min	$(5, f(5)) = (5, -3)$

2. A company finds that the profit $P(x)$ where x represents thousands of units, is given by

$$P(x) = -x^3 + 9x^2 - 15x - 9$$

If the company can only make a maximum of 6000 units, what is the absolute maximum profit?

Here, * x reprsents thousands of units

* the compagny can only make a max 6 (thousands) units

* the domain of the profit is then $[0, 6]$

Now, $P'(x) = -3x^2 + 18x - 15 = -3(x^2 - 6x + 5)$

$= -3(x-5)(x-1)$

$P'(x) = 0 \Rightarrow \quad x = 5 \in [0, 6]$ and $x = 1 \in [0, 6]$

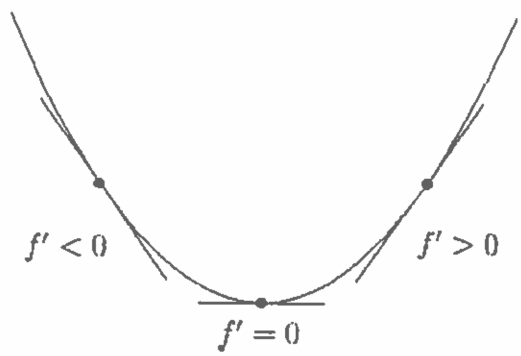
x	$P(x)$		
0	$P(0) = -9$		
1	$P(1) = -16$	← abs min	at $(1, -16)$
5	$P(5) = 16$	← abs max	at $(5, 16)$
6	$P(6) = 9$		

We close out this section with a helpful theorem.

Theorem 1.2 *If f is a continuous function on an interval and has only one critical value in that interval, then a relative max or min is also an absolute max or min.*

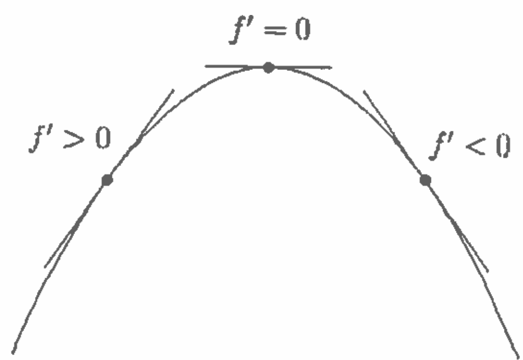
2 Concavity and the Second-Derivative Test

Intuition: a curve is **concave up** on an interval I if it looks like  on I. It is concave down on I if it looks like . We need a more precise definition.



The slope $m = f'(x)$ increases once x increases. This means that $f'(x)$ is increasing function ($f''(x) > 0$)

So f' is increasing on this interval.



The slope $m = f'(x)$ decreases once x increases. This means that $f'(x)$ is deacreasing function ($f''(x) < 0$)

And f' is decreasing on this interval. These two suggest the following precise definition of concavity.

Definition 2.1 *If f is a function, we say f is* **concave up** *on I if f' is increasing on I. We say that f is* **concave down** *on I if f' is decreasing on I.*

Theorem 2.1 *Let f be a function. Then*

1. *If $f''(x) > 0$ for all x in I, then f is concave up on I.*
2. *If $f''(x) < 0$ for all x in I, then f is concave down on I.*

<u>Examples:</u>

1. Find the intervals of concavity for $f(x) = \frac{x^3}{3} - 2x^2$. $D_f = \mathbb{R}$

$f'(x) = x^2 - 4x$

$f''(x) = 2x - 4$. $f''(x) = 0 \Rightarrow x = 2$.

x	$-\infty$		2		∞
$f''(x)$		$-$	0	$+$	
Concavity of f		∩		∪	

* f is concave up on $(2, \infty)$
* f is concave down on $(-\infty, 2)$

2. Find the intervals of concavity for $f(x) = x^4 + 2x^3 - 12x^2 + x + 5$. $D_f = \mathbb{R}$

$f'(x) = 4x^3 + 6x^2 - 24x + 1$

$f''(x) = 12x^2 - 12x - 24 = 12(x+2)(x-1)$

$f''(x) = 0 \Rightarrow x = -2$ or $x = 1$

x	$-\infty$	-3	-2	0	1	3	∞
Test points for $f''(x)$		$f''(-3) > 0$		$f''(0) < 0$		$f''(3) > 0$	
$f''(x)$		$+$	0	$-$	0	$+$	
concavity of f		∪		∩		∪	

f is concave up on $(-\infty, -2) \cup (1, \infty)$

f is concave down on $(-2, 1)$

Note that the first graph above has a local min and the second has a local max. This leads to the following

Theorem 2.2 (The Second-Derivative Test): *Suppose that c is a critical point for the function f. Then*

1. *If $f''(c) > 0$, then f has a local min at c.*
2. *If $f''(c) < 0$, then f has a local max at c.*
3. *If $f''(c) = 0$ or $f''(c)$ does not exist, then the test fails.*

Note that the second derivative test is easier to use, but sometimes fails. The first derivative test always works.

Example: Consider the following function $f(x) = x^3 - 6x^2 - 15x + 3$

1. Find all possible critical points
2. Use the Second-Derivative Test to classify all possible critical points.

$D_f = \mathbb{R}$.

1.) $f'(x) = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5)$
$= 3(x-5)(x+1) = 0$

$f'(x) = 0 \Rightarrow x = 5 \in D_f$ and $x = -1 \in D_f$

The critical points are: $x = 5$ and $x = -1$

2) Second derivative Test

$f''(x) = (f'(x))' = (3x^2 - 4x - 5)' = 6x - 12$

at $x = 5$ $f''(5) = 6(5) - 12 = 18 > 0$

so, f admits a local min at $x = 5$ and the min of f is $f(5)$. Note: the point $(5, f(5))$ is a point of a local min.

at $x = -1$ $f''(-1) = -18 < 0 \Rightarrow f$ admits a local max at $x = -1$. and the max of f is $f(-1)$. Note that: the point $(-1, f(-1))$ is a point of a local max.

Point of inflection

Theorem 2.3 *Let c be a point from the domain of the function f. We say that c is* **point of inflection** *if its satisfies the following two conditions:*

i) $f''(c) = 0$;

ii) *at point c, the concavity of $f(x)$ changes from up to down or vice-versa.*

Examples:

1. Find the intervals of concavity and points of inflection for $f(x) = x^4 + x^3 - 3x^2 + 1$.

   solution: We calculate the second derivative

$$f'(x) = 4x^3 + 3x^2 - 6x$$

$$f''(x) = 12x^2 + 6x - 6 = 6(2x - 1)(x + 1)$$

Thus $x = \frac{1}{2}, -1$ are our possible inflection points. Thus we can use test points as before to produce the following diagram:

So,

- f is concave up on $(-\infty, -1) \cup (\frac{1}{2}, \infty)$;
- f is concave down on $(-1, \frac{1}{2})$;
- f has an inflection point at $x = -1$, since $f''(-1) = 0$ and the concavity of f changes at -1;
- f has an inflection point at $x = \frac{1}{2}$, since $f''(\frac{1}{2}) = 0$ and the concavity of f changes at $\frac{1}{2}$.

Remark: Note that just because $f''(c) = 0$ does not mean c is an inflection point. See the following example.

2. Show that $f(x) = x^4$ has a point c where the second derivative is zero but $x = c$ is <u>not</u> an inflection point).

$$f'(x) = 4x^3$$

$$f''(x) = 12x^2$$

$$f''(x) = 0 \Rightarrow 12x^2 = 0 \Rightarrow x = 0$$

x	$-\infty$		0		∞
$f''(x)$		$+$	0	$+$	
concavity of $f(x)$		$\cup$		$\cup$	

$f''(x) = 12x^2 \geq 0$

at $x = 0$, we have $f''(x) = 0$. However there is no change of the concavity of f at $x = 0$. Hence, $x = 0$ is not an inflection point for f.