MAT1339 B - Winter 2018

Lecture - 8

1 Optimization: Minimization and Maximization problems

We can use calculus to solve real-world problems!

Example: (maximization problem)

A manufacturer wants to design an open box with a square base and a surface area of 108 in^2. What dimensions will maximize volume? What is the maximum volume?

* the volume of this open box is

$$V = x \cdot x \cdot h = x^2 h \quad (1)$$

* The total area is

$$A = x \cdot x + x \cdot h + x \cdot h + x \cdot h + x \cdot h$$
$$= x^2 + 4x \cdot h = 108 \quad (2)$$

From (2) $4xh = 108 - x^2 \Rightarrow h = \dfrac{108 - x^2}{4x} = \dfrac{27}{x} - \dfrac{x}{4}$ (3)

using (3) in (1) we get:

$$V(x) = x^2\left(\frac{27}{x} - \frac{x}{4}\right) = 27x - \frac{x^3}{3}$$

Now, the volume is a function of only one variable x. Next, we find critical points:

* First, the domain of the dimensions is $(0, \infty)$

Now, $V'(x) = \left(27x - \dfrac{x^3}{3}\right)' = 27 - \dfrac{3}{4}x^2$ (4)

Solve $V'(x) = 0 \Rightarrow x^2 = 27\left(\frac{4}{3}\right) = 36$

$$\Rightarrow x = \pm 6$$

we accept $x = 6 \in (0, \infty)$. However, $x = -6 \notin (0, \infty)$

we have only one critical point

Example: (maximization problem.......)

To determine that the critical point $x = 6$ is a local max, we will use the second derivative test.

$V''(x) = (27 - \frac{3}{4}x^2)' = -\frac{6}{4}x$. So that at $x = 6$

$V''(6) = -\frac{6}{4}(6) = -\frac{36}{4} = -9 < 0 \Rightarrow x = 6$ is a local max.

Finally, since $x = 6$ is the only one critical point, we conclude that $x = 6$ is an absolute max.

Next, we find the value of h: From ③ we have

$h = \frac{108 - x^2}{4x} = \frac{108 - (6)^2}{4(6)} = 3.$

Therefore, the dimensions that maximize the volume are: $x = 6$ in and $h = 3$ in.

the maximum volume is then From ① $V = x^2 \cdot h = 6^2 \cdot (3)$
$= 108 \text{ in}^3$

Important remarks:

when doing these types of problems you have to verify that your solution is an absolute max or absolute min. There are three ways of doing this:

1. If the function is a simple one like a parabola you may be able to verify it by inspection. For example, parabolas have the form $f(x) = ax^2 + bx + c$ and have only one critical point. If $a > 0$ it is an absolute min and if $a < 0$ it is an absolute max.

2. We can use this theorem: If f is continuous on a closed interval $[a, b]$, then f has both an absolute max and an absolute min on $[a, b]$. To find them,

   (a) Evaluate f at each critical point in $[a, b]$.

   (b) Evaluate f at the endpoints a and b.

   (c) The least of these values is the absolute min and the greatest is the absolute max.

3. We can use this theorem: If f is a continuous function on an interval and has only one critical value in that interval, then a relative max or min is also an absolute max or min.

Average Cost

Due to the nature of cost functions, the minimum cost usually corresponds to making 0 units. This isn't very useful information, as it is obvious that if we make no units we will spend the least money. More informative would be to find the production level that minimizes the **cost per unit**, or **average cost**:

$$\bar{C}(x) = \frac{C(x)}{x}$$

Example: (minimization problem)

Suppose that our cost function is $C(x) = 800 + 0.04x + 0.0002x^2$. Find the production level that minimizes the average cost.

In this case the average cost is

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{800}{x} + 0.04 + 0.0002x$$

1. We find the critical points

$$\bar{C}'(x) = \frac{-800}{x^2} + 0.0002 = 0 \Rightarrow x^2 = \frac{800}{0.0002}$$
$$= 4{,}000{,}000$$

Since we are only considering x in $(0, \infty)$, we have one critical point $x = \sqrt{4{,}000{,}000} = 2000$.

2.) We use the second derivative Test.

$$\bar{C}''(x) = \left(\frac{-800}{x^2} + 0.0002\right)' = \frac{1600}{x^3}$$

$$\bar{C}''(2000) = \frac{1600}{(2000)^3} > 0$$

So $x = 2000$ is a local min. Since $x = 2000$ is the only critical point in the domain $(0, \infty)$, we conclude that $x = 2000$ is an absolute min.

3.) The min average is $\bar{C}(2000) = \frac{800}{2000} + 0.04 + 0.0002(2000)$
$= \$0.84$

2 Vertical and Horizontal Asymptotes for a rational function

Recall that

$x = c$ is a vertical asymptote of $f(x)$ if

$$\lim_{x \to c^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to c^-} f(x) = \pm\infty$$

and

$y = L$ is called a horizontal asymptote for f if

$$\lim_{x \to \infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L,$$

2.1 Vertical Asymptote for a rational function

If f is a rational function

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are both polynomials, then $x = c$ is a vertical asymptote for f if

1. $p(c) \neq 0$ and
2. $q(c) = 0$

Examples:

1. Suppose that f is defined as

$$f(x) = \frac{x+4}{x^2 - 4x} = \frac{x+4}{x(x-4)}$$

The domain of this function is $(-\infty, 0) \cup (0, 4) \cup (4, \infty)$ because the denominator is 0 when $x = 0$ or $x = 4$.

Now, set $p(x) = x + 4$ and $q(x) = x(x - 4)$

For $x = 0$, we have $p(0) = 4$ and $q(0) = 0$ means that $x = 0$ is a vertical asymptote.

For $x = 4$, we have $p(4) = 8$ and $q(4) = 0$ means that $x = 4$ is a vertical asymptote.

2.2 Horizontal asymptotes for a rational function

Suppose $f(x) = \frac{p(x)}{q(x)}$, where p and q are both polynomials. Then

- If the degree of p is less than the degree of q, then f has a horizontal asymptote at $y = 0$.
- If the degree of p is equal to the degree of q, then f has a horizontal asymptote at $y = L$ where L is the ratio of the leading coefficient of p to the leading coefficient of q.
- If the degree of p is greater than the degree of q, then the function has no horizontal asymptotes.

Examples: Find all possible horizontal asymptotes for the following rational functions

1. $f(x) = \dfrac{2x+1}{4x^2+5}$.

   $p(x) = 2x+1$ and $q(x) = 4x^2+5$

   $\deg(P) = 1 < \deg(q) = 2 \Rightarrow y = 0$ is a H.A

2. $f(x) = \dfrac{3x^2+2x+10}{-2x^2+x-4}$

   $P(x) = 3x^2+2x+10$ and $q(x) = -2x^2+x-4$

   $\deg(P) = \deg(q) = 2 \Rightarrow y = \frac{a_n}{b_n} = \frac{3}{-2}$ is a H.A

3. $f(x) = \dfrac{3x^7+x^5+x+1}{-2x^4+x^3+x-4}$

   $\deg(P) = 7 > \deg(q) = 4$

   $\Rightarrow f(x)$ has no H.A.

3. Curve Sketching

When asked to graph a function, your graph should take into account the following information:

1. domain,
2. x-intercepts and y-intercept,
3. continuity (ie places where it is not continuous),
4. differentiability (ie places where it isn't differentiable),
5. local max's and min's,
6. intervals of increase and decrease,
7. intervals of concavity,
8. points of inflection,
9. asymptotes.

Examples 1: Sketch the graph of $g(x) = \frac{x+2}{x-1}$.

* Domain $= \mathbb{R}\setminus\{1\} = (-\infty, 1) \cup (1, \infty)$ since $x \neq 1$

* x-intercept $f(x) = 0 \Rightarrow \frac{x+2}{x-1} = 0 \Rightarrow x+2 = 0 \Rightarrow x = -2$
  So that $f(x)$ crosses x-axis at $(-2, 0)$

* y-intercept set $x = 0$ and compute $f(0) = \frac{0+2}{0-1} = -2$
  so that $f(x)$ crosses y-axis at $(0, -2)$

* Vertical and Horizontal asymptotes
  V.A since $f(x)$ is a rational function
  set $p(x) = x+2$ and $q(x) = x-1$
  at $x = 1$ $p(1) = 1+2 = 3 \neq 0$ and $q(1) = 0$
  So that $f(x)$ admits a V.A $\boxed{x = 1}$
  Now, $\lim_{x \to 1^+} f(x) = \frac{3}{0^+} = +\infty$ and $\lim_{x \to 1^-} f(x) = \frac{3}{0^-} = -\infty$

H. A   since $f(x)$ is a rational function

$$\lim_{x \to +\infty} f(x) = \lim_{x \to \infty} \frac{x+2}{x-1} = \frac{1}{1} = 1$$

Here $\deg(P) = \deg(q)$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x+2}{x-1} = \frac{1}{1} = 1$$

Hence, $\boxed{y = 1}$ is a H A

* First derivative test

$$f'(x) = \left(\frac{x+2}{x-1}\right)' = \frac{(x+2)'(x-1) - (x+2)(x+1)'}{(x-1)^2}$$

$$= \frac{(1)(x-1) - (x+2)(1)}{(x-1)^2}$$

$$= \frac{x-1-x-2}{(x-1)^2} = \frac{-3}{(x-1)^2} < 0$$

	$-\infty$	1	∞
$f'(x)$	$-$	||	$-$
$f(x)$	$\searrow$	||	$\searrow$

* There is no critical point
* f is decrasing on $(-\infty, 1) \cup (1, \infty)$
* There is no local max
* there is no local min

* <u>Concavity of f(x)</u>

since $f'(x) = \frac{-3}{(x-1)^2} = -3\,(x-1)^{-2}$

so, $f''(x) = \left(-3(x-1)^{-2}\right)' = -3(-2)\,(x-1)'\,(x-1)^{-2-1}$

$= 6\,(1)\,(x-1)^{-3} = \frac{6}{(x-1)^3} \neq 0$

There is no inflexion point since $f''(x) \neq 0 \quad \forall x \in D_f$

x	$-\infty$	1 ∞
Test point	$f''(0) = \frac{6}{-1} = -6 < 0$	$f''(2) = \frac{6}{1} = 6 > 0$
$f''(x)$	$-$	$+$
concavity of $f(x)$	∩	∪

Domain = $(-\infty, 1) \cup (1, \infty)$

$x = 1 \notin$ Domain

* f is concave down on $(-\infty, 1)$
* f is concave up on $(1, \infty)$
* There is no inflexion point since $f''(x) = \frac{6}{(x-1)^3} \neq 0$

<u>Graph of f(x)</u>