MAT1339 B - Winter 2018

Lecture - 9

1 Exponential and Logarithmic Functions

1.1 Exponential Functions and the Natural Base e

Definition 1.1 *If $a > 0$ and $a \neq 1$, then the exponential function with base a is given by $f(x) = a^x$.*

An important special case is when $a = e \approx 2.71828\ldots$, an irrational number.

Properties of Exponents

Let $a, b > 0$. Then,

1. $a^0 = 1$
2. $a^x a^y = a^{x+y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$
5. $\frac{a^x}{a^y} = a^{x-y}$
6. $(\frac{a}{b})^x = \frac{a^x}{b^x}$
7. $a^{-x} = 1/a^x$

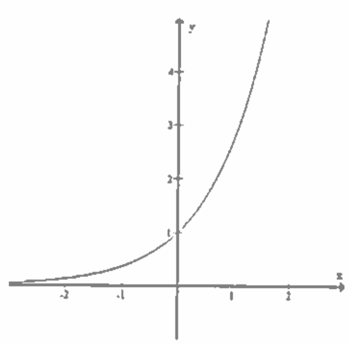
Graphs:

1. Exponential: e^x

$D_f = \mathbb{R}$, $Im f = \mathbb{R}^+$

$\lim_{x \to +\infty} e^x = +\infty$

$\lim_{x \to -\infty} e^x = 0$



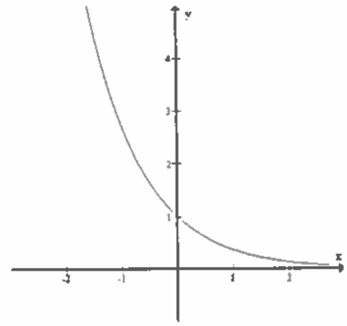
e^x crosses y-axis at $(0,1)$: y-intercep

2. Exponential: e^{-x}

$D_f = \mathbb{R}$, $\mathrm{Im} f = \mathbb{R}^+$

$\lim_{x \to +\infty} e^{-x} = 0$

$\lim_{x \to -\infty} e^{-x} = +\infty$

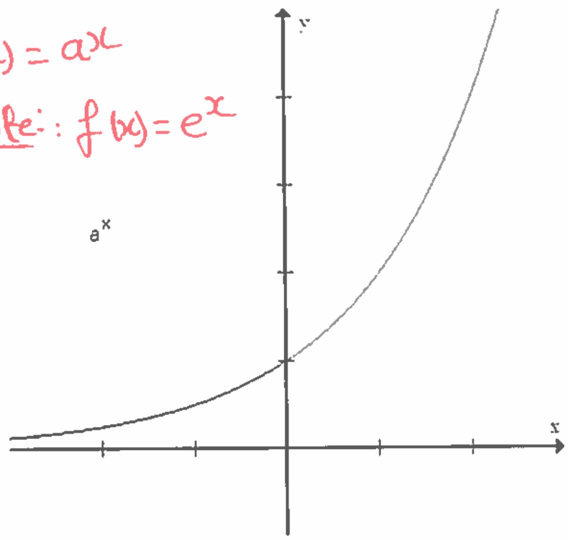


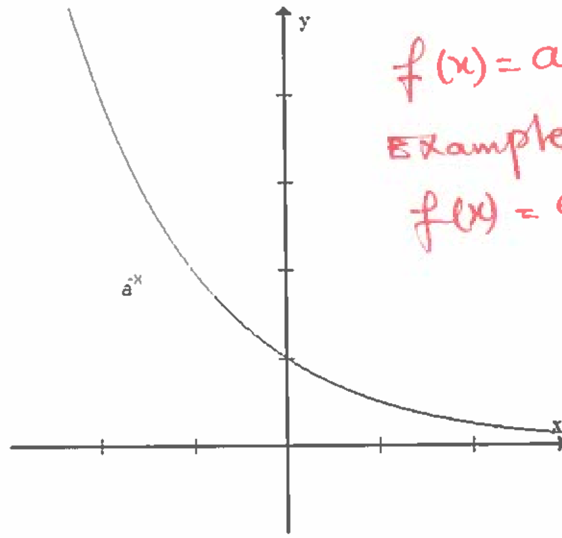
e^{-x} crosses y-axis at $(0,1)$: y-intercept

3. More generally, if $a > 1$, for example $a = e = 2.71828$

$f(x) = a^x$

Example: $f(x) = e^x$





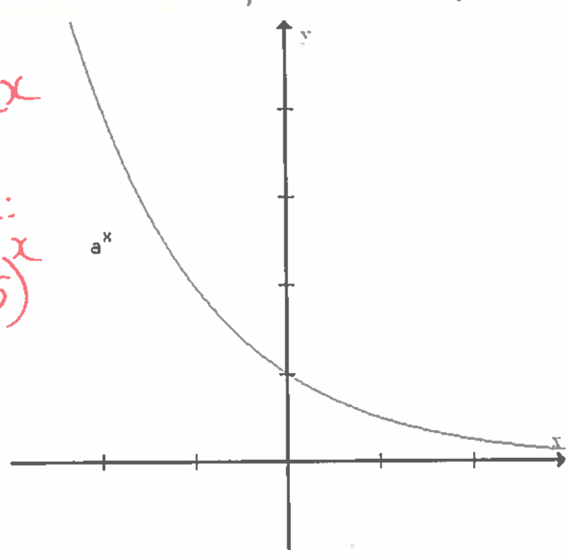
$f(x) = a^{-x}$

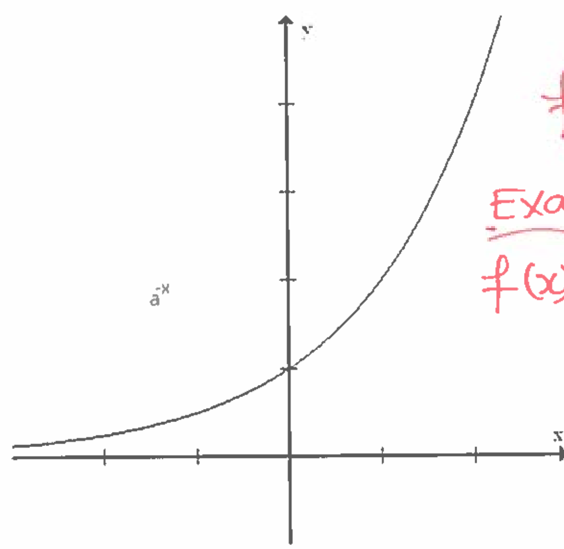
Example:
$f(x) = e^{-x}$

If $0 < a < 1$, for example $a = 0.5$

$f(x) = a^x$

Example:
$f(x) = (0.5)^x$



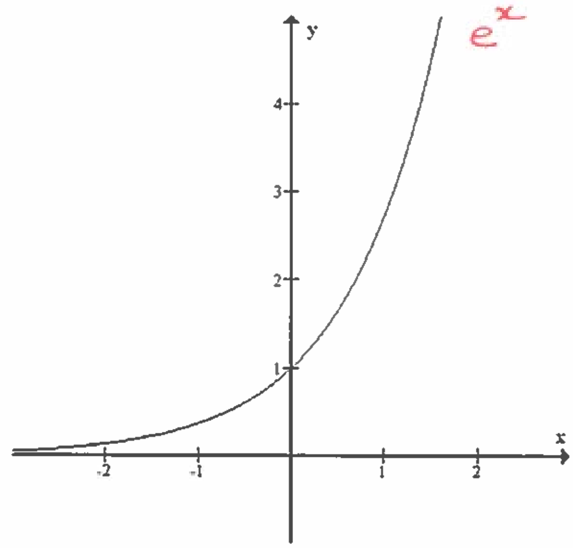


$f(x) = a^{-x}$

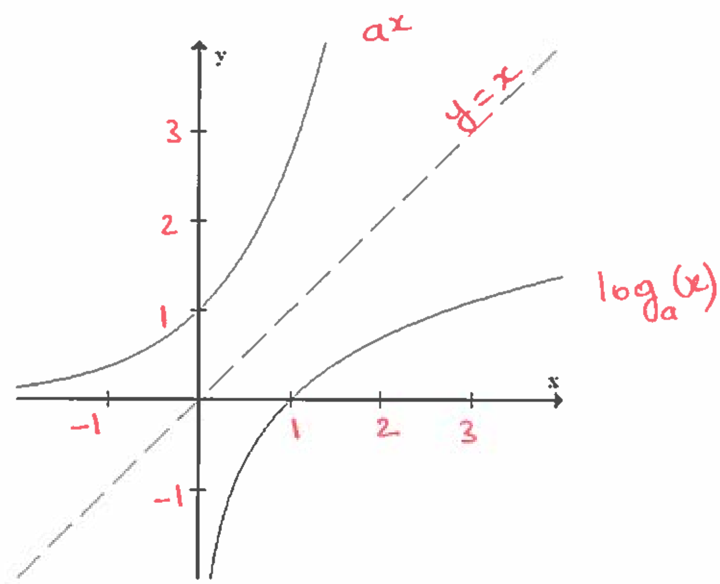
Example
$f(x) = (0.5)^{-x}$

1.2 Logarithms

Note the graph of e^x



More generally, for any $a > 1$ the graph of a^x and its inverse look like this.



If $f(x) = a^x$, then we define the inverse function f^{-1} to be the **logarithm with base** a, and write

$$f^{-1}(x) = \log_a(x)$$

Note that, since the image of a^x is only the positive numbers, the domain of $\log_a(x)$ is all positive real numbers. The key property is:

$\log_a(x): (0,\infty) \rightarrow \mathbb{R}$

$$\boxed{\log_a x = b \iff a^b = x}$$

Examples:

$\log_{10} 10 = 1$	$10^{1} = 10$
$\log_5 25 = 2$	$5^{2} = 25$
$\log_4 \frac{1}{2} = -\frac{1}{2}$	$4^{-\frac{1}{2}} = \frac{1}{2}$
$\log_5 \frac{1}{125} = -3$	$5^{-3} = \frac{1}{125}$
↑	↑
log equation	corresponding exponential equation

Log Rules

1. $\log_a(xy) = \log_a(x) + \log_a(y)$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^y) = y\log_a(x)$

Natural logarithm function

The most important case of logs is when $a = e$. Log base e has a special name, in fact we define $\log_e x = \ln(x)$. So that

$$\boxed{\ln(e^x) = x \text{ and } e^{\ln(x)} = x}.$$

For more general case, if $g(x) > 0$ then: $\ln\left(e^{g(x)}\right) = e^{\ln(g(x))} = g(x)$

EX: $\ln\left(e^{(x^2+1)}\right) = x^2+1$

The function $\ln(x)$ is known as the **natural logarithm function**,

Note that

$$\log_a(x) = \frac{\ln(x)}{\ln(a)},$$

Example

$$\log_e(x) = \frac{\ln(x)}{\ln(e)} = \frac{\ln(x)}{1} = \ln(x)$$

ln rules

1. $\ln(1) = 0$
1. $\ln(e) = 1$
1. $\ln(xy) = \ln(x) + \ln(y)$
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
3. $\ln(x^y) = y\ln(x)$

Calculations:

$$e^{3\ln(x)} = e^{\ln(x^3)} = x^3$$

$$\ln\left(\frac{1}{e}\right) = \ln(1) - \ln(e) = 0 - 1 = -1$$

Rewrite the following:

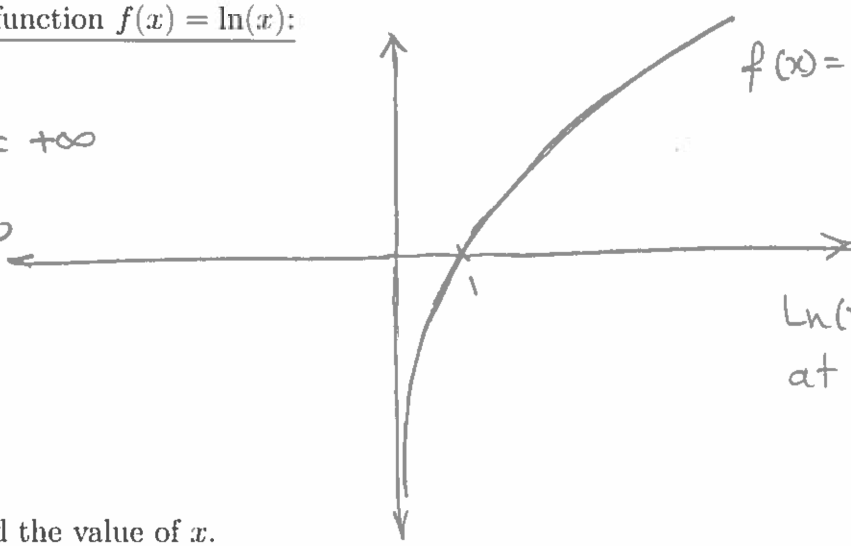
$$\ln\left(\frac{xy}{z}\right) = \mathrm{Ln}(xy) - \mathrm{Ln}(z) = \mathrm{Ln}(x) + \mathrm{Ln}(y) - \mathrm{Ln}(z)$$

Graph of the function $f(x) = \ln(x)$:

$D_f = (0, \infty)$

$\lim_{x \to +\infty} \ln(x) = +\infty$

$\lim_{x \to -\infty} \mathrm{Ln}(x) = 0$



$f(x) = \mathrm{Ln}(x)$

$\mathrm{Ln}(x)$ crosses x-axis at $(1, 0)$: x-intercept

Example: Find the value of x.

1. $4^{2x+1} = 64^x$

$\Rightarrow 4^{2x+1} = (4^3)^x = 4^{3x}$

$\Rightarrow 2x + 1 = 3x$

$\Rightarrow 3x - 2x = 1$

$\Rightarrow x = 1$

2. $e^{3x} = 2$

$\mathrm{Ln}(e^{3x}) = \mathrm{Ln}(2)$

$\Rightarrow 3x = \mathrm{Ln}(2) \Rightarrow x = \frac{\mathrm{Ln}(2)}{3}$

3. $\log x - \log 2 = \log 5$

First, $x > 0$. Now, $\log(x) - \log(2) = \log(5)$

$\Rightarrow \log\left(\frac{x}{2}\right) = \log(5)$

$\Rightarrow \frac{x}{2} = 5 \Rightarrow x = 10$

4. $\ln x + \ln(2x) = 0$

First, $x > 0$ and $2x \geq 0$ so $x > 0$.

Now, $\mathrm{Ln}(x) + \mathrm{Ln}(2x) = 0 \Rightarrow \mathrm{Ln}(x \cdot 2x) = 0 \Rightarrow \mathrm{Ln}(2x^2) = \mathrm{Ln}(1)$

$\Rightarrow 2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$, since $x > 0$

we accept $x = \frac{1}{\sqrt{2}}$

2 Derivatives of exponential and logarithmic functions

2.1 Derivative of exponential functions

for $a > 0$

- $(a^x)' = a^x \ln(a)$
- $(a^{g(x)})' = a^{g(x)} g'(x) \ln(a)$ (for more general case)

Example Find

1. $(3^x)' = 3^x \cdot \ln(a)$
2. $(9^{-x+2})' = 9^{-x+2} \cdot (-x+2) \cdot \ln(9) = 9^{-x+2} (-1) \ln(9)$
3. $(2^{x^2})' = 2^{x^2} (x^2)' \ln(2) = 2^{x^2} (2x) \ln(2)$
4. $(8^{\sqrt{2x+1}})' = 8^{\sqrt{2x+1}} \cdot (\sqrt{2x+1})' \ln(8) = 8^{\sqrt{2x+1}} \cdot \left(\frac{1}{2} (2) (2x+1)^{-\frac{1}{2}}\right) \ln(8)$

Special case $a = e$

- $(e^x)' = e^x$
- $(e^{g(x)})' = e^{g(x)} g'(x)$ (for more general case)

Example Find

1. $(5e^x)' = 5(e^x)' = 5e^x$
2. $(e^{7x})' = e^{7x} (7x)' = e^{7x} (7)$
3. $(e^{x^2})' = e^{x^2} (x^2)' = e^{x^2} (2x)$
4. $(e^{\sqrt{2x+1}})' = e^{\sqrt{2x+1}} \cdot \left((2x+1)^{1/2}\right)' = e^{\sqrt{2x+1}} \cdot \left(\frac{1}{2} (2) (2x+1)^{-\frac{1}{2}}\right)$

2.2 Derivative of logarithmic functions

- $[\ln(x)]' = \frac{1}{x}, \quad x > 0.$
- $[\ln(g(x))]' = \frac{g'(x)}{g(x)}$ (for more general case). Here, $g(x)$ is a positive function.

Examples: Find the derivative of the following functions

1. $y = 5\ln(x)$

$$y' = 5(\ln(x))'$$
$$= 5 \cdot \frac{1}{x}$$
$$= \frac{5}{x}$$

2. $y = \ln(x^2+1)$

$$y' = \ln'(x^2+1) = \frac{(x^2+1)'}{x^2+1} = \frac{2x}{x^2+1}$$

3. $\ln\left(\sqrt{x^3+7x}\right) = \ln\left(\left(x^3+7x\right)^{1/2}\right) = \frac{1}{2}\ln\left(x^3+7x\right)$

$$\Rightarrow \ln'\left(\sqrt{x^3+7x}\right) = \frac{1}{2}\ln'\left(x^3+7x\right) = \frac{1}{2}\frac{(x^3+7x)'}{x^3+7x}$$
$$= \frac{1}{2}\frac{3x^2+7}{x^3+7x}$$

Example Suppose $y = \frac{\ln(x)}{x}$. Find the equation of the tangent line at $x = e$.

$$\begin{aligned} y' &= \frac{\left(\frac{1}{x}\right)x - \ln(x)(1)}{x^2} \\ &= \frac{1-\ln(x)}{x^2} \end{aligned}$$

Plugging in $x = e$ gives us $\frac{1-\ln(e)}{e^2} = 0$. So the slope is 0, and our equation so far is $y = 0x + b$. Plugging in the point $\left(e, \frac{1}{e}\right)$ gives us $b = \frac{1}{e}$, so the equation of the tangent line is $y = \frac{1}{e}$.