

1 Exponential and Logarithmic Functions with applications

Examples Determine the derivative of the following functions

1. $y = xe^x$

$$\begin{aligned} y' &= (xe^x)' = (x)'e^x + x(e^x)' \\ &= 1 \cdot e^x + x(e^x) \\ &= e^x + xe^x \end{aligned}$$

2. $y = e^{x^2} \ln(3x^2 - 1)$

$$\begin{aligned} y' &= (e^{x^2})' \ln(3x^2 - 1) + e^{x^2} (\ln(3x^2 - 1))' \\ &= (x^2)' e^{x^2} \ln(3x^2 - 1) + e^{x^2} \frac{(3x^2 - 1)'}{3x^2 - 1} \\ &= 2xe^{x^2} \ln(3x^2 - 1) + e^{x^2} \frac{6x}{3x^2 - 1} \end{aligned}$$

3. $y = x^2 10^x$

$$\begin{aligned} y' &= (x^2)' 10^x + x^2 \cdot (10^x)' \\ &= 2x 10^x + x^2 10^x \ln(10) \end{aligned}$$

4. $f(x) = xe^{x^2} + 2e^{-3x}$

$$\begin{aligned} f'(x) &= (x)'e^{x^2} + x(e^{x^2})' + 2(e^{-3x})' \\ &= 1e^{x^2} + x(x^2)'e^{x^2} + 2(-3x)'e^{-3x} \\ &= e^{x^2} + x(2x)e^{x^2} + 2(-3)e^{-3x} = e^{x^2} + 2x^2e^{x^2} - 6e^{-3x} \end{aligned}$$

5. $y = \ln\left(\frac{e^x}{\sqrt{x}}\right) = \ln(e^x) - \ln(\sqrt{x}) = x - \ln(x^{1/2}) = x - \frac{1}{2} \ln(x)$

Now, $y' = \left(\ln\left(\frac{e^x}{\sqrt{x}}\right)\right)' = \left(x - \frac{1}{2} \ln(x)\right)' = 1 - \frac{1}{2} \cdot \frac{1}{x} = 1 - \frac{1}{2x}$

Examples

Here, $Df = \mathbb{R}$ since $x^2 + 2x + 2 > 0 \quad \forall x \in \mathbb{R}$

1. Determine the point of the function $f(x) = x + \ln(x^2 + 2x + 2)$ for which the slope of the tangent line at this point is 0.

* First, the slope of the tangent line at $x \in Df$ is

$$f'(x) = (x + \ln(x^2 + 2x + 2))' = 1 + \frac{(x^2 + 2x + 2)'}{x^2 + 2x + 2} = 1 + \frac{2x + 2}{x^2 + 2x + 2}$$
$$\Rightarrow f'(x) = \frac{x^2 + 2x + 2 + 2x + 2}{x^2 + 2x + 2} = \frac{x^2 + 4x + 4}{x^2 + 2x + 2}$$

We need to find $x \in Df$ such that $f'(x) = 0 \Rightarrow$
 $x^2 + 4x + 4 = 0 \Rightarrow (x + 2)^2 = 0 \Rightarrow x = -2$

Hence, if $x = -2$ then $f'(-2) = 0$

2. Determine the equation of the tangent line to the function $f(x) = \frac{\ln x}{3x}$, at $(1, f(1))$.

$$y = f'(1)(x - 1) + f(1) \quad (\text{Equation of the tangent line})$$

We need to evaluate $f'(1)$ and $f(1)$

$$f'(x) = \left(\frac{\ln(x)}{3x} \right)' = \frac{\ln'(x) \cdot 3x - \ln(x) \cdot (3x)'}{(3x)^2}$$

$$= \frac{\frac{1}{x} \cdot 3x - \ln(x) \cdot 3}{(3x)^2}$$

$$= \frac{3 - 3 \ln(x)}{9x^2} = \frac{1 - \ln(x)}{3x}$$

$$\Rightarrow f'(1) = \frac{1 - \ln(1)}{3(1)} = \frac{1 - 0}{3} = \frac{1}{3}$$

$$f(1) = \frac{\ln(1)}{3(1)} = \frac{0}{3} = 0. \quad \text{So that}$$

$$y = f'(1)(x - 1) + f(1)$$

$$= \frac{1}{3}(x - 1) + 0 = \frac{1}{3}x - \frac{1}{3}$$

3. Let $f(x) = (x^2 - 15)e^x$ be a function. Determine the local max and the local min of f

1.) $Df = \mathbb{R} = (-\infty, \infty)$

2.) critical points: $f'(x) = (x^2 - 15)'e^x + (x^2 - 15)(e^x)'$
 $= 2xe^x + (x^2 - 15)e^x$
 $= (2x + x^2 - 15)e^x = (x^2 + 2x - 15)e^x$

$f'(x) = 0 \Rightarrow x^2 + 2x - 15 = 0$ since $\forall x \in \mathbb{R} e^x > 0$
 so, $x^2 + 2x - 15 = 0 \Rightarrow (x - 3)(x + 5) = 0 \Rightarrow x = 3$ or $x = -5$

First derivative Test

	$-\infty$	-5	3	∞
Test points	$f'(-6) > 0$	$f'(0) < 0$	$f'(4) > 0$	
$f'(x)$	+	0	-	+
$f(x)$		↘ $f(-5)$ ↗		
			↘ $f(3)$ ↗	

at $x = -5$: f admits a local max. $f(-5)$ is the max of f

at $x = 3$ f admits a local min $f(3)$ is the min of f

4. Let $f(x) = e^x(x^2 - 3)$ be a function where $x \in [-5, 2]$. Determine the ~~local~~ absolute max and the ~~local~~ absolute min of f

1.) $Df = \mathbb{R} = (-\infty, \infty)$

2.) critical points: $f'(x) = (e^x)'(x^2 - 3) + e^x(x^2 - 3)'$
 $= e^x(x^2 - 3) + e^x(2x)$
 $= e^x(x^2 - 3 + 2x)$
 $= (x^2 + 2x - 3)e^x$

$f'(x) = 0 \Rightarrow x^2 + 2x - 3 = 0 \Rightarrow (x + 3)(x - 1) = 0$
 $\Rightarrow x = -3$ or $x = 1$

since, $x = -3 \in [-5, 2] \Rightarrow x = -3$ is a c.p

since, $x = 1 \in [-5, 2] \Rightarrow x = 1$ is a c.p

x	$f(x)$
-5	$f(-5) = 0.1482$
-3	$f(-3) = 0.2987$
1	$f(1) = -5.436 \leftarrow \text{abs min}$
2	$f(2) = 7.389 \leftarrow \text{abs max}$

2 Applications: Exponential Growth and Decay

Example Bacterial growth is a perfect example - the more bacteria there are in a colony, the faster the colony grows due to cell division. Cases like this are modelled by the equation

$$P(t) = P_0 e^{kt}$$

where

- $P(t)$ is the number of bacteria at time t
- P_0 is the initial population (initial value)
- k is the constant of proportionality.

Note that,

- if $k > 0$, we say that y **grows exponentially**
- if $k < 0$ we say that y **decays exponentially**

Remark

- The reason we say P_0 is the initial value is that when $t = 0$, $P(0) = P_0 e^0 = P_0$.

Example: A bacterial colony's population is modelled by the function $P(t) = 50e^{0.5t}$, where P is the number of bacteria after t days.

1. What is the bacterial population after 3 days?

at $t = 3$, we have $P(3) = 50 e^{0.5(3)} =$

2. How long will it take for the population to reach 10 times its initial level?

We need to find t such that $P(t) = 3P_0$
since, $P_0 = P(0) = 50 e^{0.5(0)} = 50$. So, we solve

$$P(t) = 3(50) = 150 \Rightarrow 50 e^{0.5t} = 150$$

$$\Rightarrow e^{0.5t} = \frac{150}{50} = 3$$

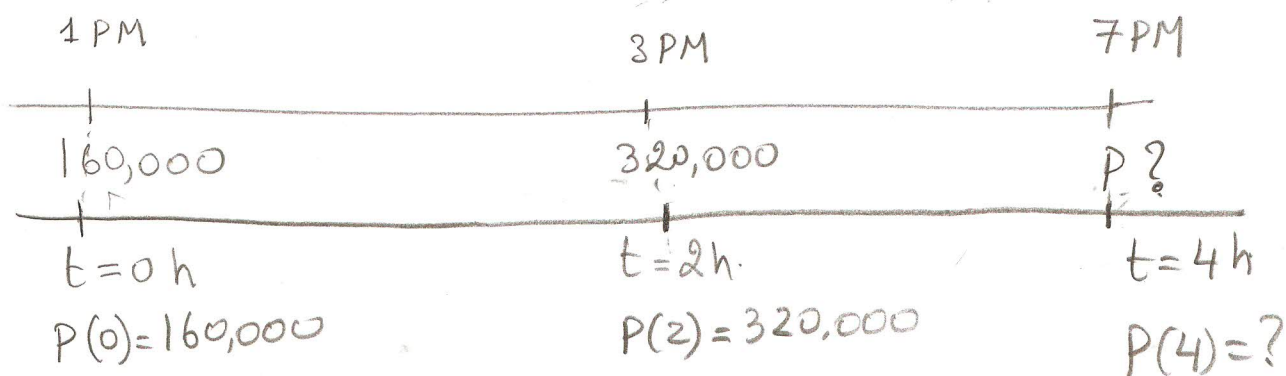
$$\Rightarrow \ln(e^{0.5t}) = \ln(3) \Rightarrow 0.5t = \ln(3)$$

$$\text{hence, } t = \frac{\ln(3)}{0.5}$$

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Examples:

1. The number of bacteria in a certain culture is 160,000 at 1pm. At 3pm, the count is 320,000. Assuming the population is growing exponentially, find the population at 7pm.



$$P(t) = P_0 e^{kt} = 160,000 e^{kt}$$

First, we need to find the value of k .

Recall that, at $t=2h$ $P(2) = 320,000$

$$\text{So, } P(2) = 160,000 e^{k(2)} = 320,000$$

$$\Rightarrow e^{2k} = \frac{320,000}{160,000} = 2$$

$$\ln(e^{2k}) = \ln(2)$$

$$\Rightarrow 2k = \ln(2) \Rightarrow k = \frac{\ln(2)}{2}$$

$$\text{So, } P(t) = 160,000 e^{kt} = 160,000 e^{\frac{\ln(2)}{2} \cdot t}$$

Now, at $t=4h$ the population is

$$\begin{aligned} P(4) &= 160,000 e^{\frac{\ln(2)}{2} \cdot 4} = 160,000 e^{\ln(2) \cdot 2} \\ &= 160,000 e^{\ln(2^2)} = 4 (160,000) \\ &= 640,000 \end{aligned}$$

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2. The number of rabbits on an island grows exponentially and takes 1 year to double. How long does it take to triple?

solution: We again assume that $P(t) = P_0 e^{kt}$. We are given that at $t = 1$, $P(t)$ will be double the initial population, ie $2P_0$. Thus

$$2P_0 = P_0 e^{k(1)}$$

$$2 = e^k$$

$$k = \ln(2)$$

Thus $P(t) = P_0 e^{\ln(2)t}$. We want t such that $P(t) = 3P_0$. Thus we solve

$$3P_0 = P_0 e^{\ln(2)t}$$

$$3 = e^{\ln(2)t}$$

$$\ln(3) = \ln(2)t$$

$$t = \frac{\ln(3)}{\ln(2)} \approx 1.58 \text{ years}$$