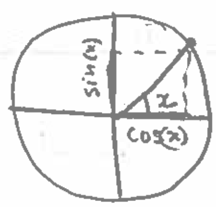
University of Ottawa
Department of math. & stat.

1. Trigonometric Functions

1.1. Introduction

Circle with radius = 1

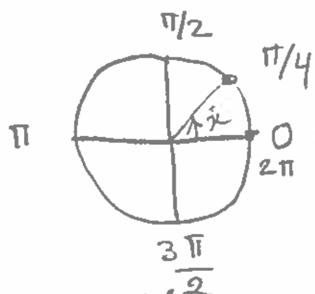


A(cos(x), sin(x))

Note that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

radian

$[0, 2\pi]$



$\pi \longrightarrow 180^\circ$

Example

* Find $\dfrac{3\pi}{4}$ in degree

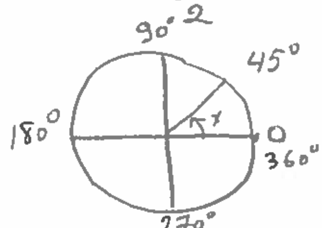
$\pi \rightarrow 180^\circ$

$\dfrac{3\pi}{4} \rightarrow x$

$\Rightarrow x = \dfrac{180^\circ}{\pi} \cdot \dfrac{3\pi}{4} = 135^\circ$

degree

$[0^\circ, 360^\circ]$



* Find 270° in radian

$\pi \longrightarrow 180^\circ$

$x \rightarrow 270^\circ$

$\Rightarrow x = \dfrac{\pi}{180} \cdot 270 = \dfrac{3}{2}\pi$

1.2. Basic formulas

$$\cos^2(x) + \sin^2(x) = 1 \quad , \quad \cos(x + 2\pi) = \cos(x) \quad , \quad \sin(x + 2\pi) = \sin(x)$$

The inverse trigonometric functions are secant, cosecante and cotangente :

$$\sec(x) = \frac{1}{\cos(x)} \quad , \quad \csc(x) = \frac{1}{\sin(x)} \quad , \quad \cot(x) = \frac{1}{\tan(x)}$$

1.3. Values of some particular angles

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\nexists$	0

The following formulas are very useful :

$$\sin(2a) = 2\sin(a)\cdot\cos(a)$$

$$\cos(2a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a) = \cos^2(a) - \sin^2(a)$$

$$\tan(2a) = \frac{2\tan(a)}{1-\tan^2(a)}$$

Exercise 1

Simplify each expression to eliminate the denominator.

a) $\frac{1-\sin^2\theta}{\cos^2\theta} = \frac{\cos^2(\theta)}{\cos^2(\theta)} = 1$

b) $\frac{\sin x}{\sin^2 x} = \frac{\cancel{\sin(x)}}{\sin(x).\cancel{\sin(x)}} = \frac{1}{\sin(x)} = \csc(x)$

c) $\frac{\sin^2 x}{\tan x} = \frac{\sin^2(x)}{\frac{\sin(x)}{\cos(x)}} = \frac{\sin^2(x)}{\sin(x)} \cdot \frac{\cos(x)}{1} = \sin(x) \cdot \cos(x)$

Exercise 2

1. **Evaluate, without using a calculator, the following expressions if they are defined**

(a) $\sec\left(\frac{\pi}{2}\right) = \frac{1}{\cos\left(\frac{\pi}{2}\right)}$ is undefined since $\cos\left(\frac{\pi}{2}\right) = 0$

(b) $\cot\left(\frac{\pi}{4}\right) = \frac{1}{\tan\left(\frac{\pi}{4}\right)} = \frac{1}{1} = 1$

(c) $\cot(\pi) = \frac{1}{\tan(\pi)}$ is undefined since $\tan(\pi) = 0$

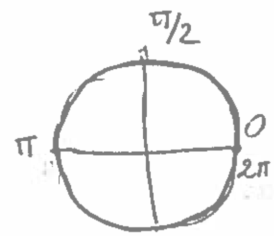
(d) $\sin\left(-\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ (Note: $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$)

2. **Solve the following equations**

(a) $1 + \sin x = 0, \quad \text{if } x \in [0, 2\pi]$

$\sin(x) = -1 = \sin\left(\frac{3\pi}{2}\right)$

$\Rightarrow x = \frac{3\pi}{2} \in [0, 2\pi]$

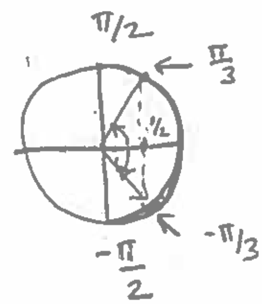


$\frac{3\pi}{2} \Leftarrow \sin(3\pi/2) = -1$

(b) $8\cos^3 x - 1 = 0, \quad \text{if } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos^3(x) = \frac{1}{8} = \left(\frac{1}{2}\right)^3 \Rightarrow \cos(x) = \frac{1}{2}$

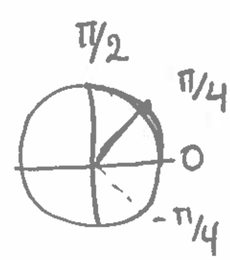
$\Rightarrow x = \frac{\pi}{3}$ or $x = -\frac{\pi}{3}$



(c) $\sin^2\theta - \cos^2\theta = 0, \quad \text{if } \theta \in \left[0, \frac{\pi}{2}\right]$

$\sin^2(\theta) = \cos^2(\theta)$

$\Rightarrow \frac{\sin^2(\theta)}{\cos^2(\theta)} = 1$



$\Rightarrow \tan^2(\theta) = 1 \Rightarrow \tan(\theta) = 1$ or $\tan(\theta) = -1$

$\Rightarrow \theta = \frac{\pi}{4} \in \left[0, \frac{\pi}{2}\right]$ or $\theta = -\frac{\pi}{4} \notin \left[0, \frac{\pi}{2}\right]$

2. Derivatives of the trigonometric functions

- **Basic functions :**

$$\cos' x = -\sin x \quad , \quad \sin' x = \cos x \quad , \quad \tan' x = \sec^2 x$$

- **The compositions, where g is a function :**

$$\cos'[g(x)] = -\sin[g(x)]\,g'(x) \quad , \quad \sin'[g(x)] = \cos[g(x)]\,g'(x) \quad , \quad \tan'[g(x)] = \sec^2[g(x)]\,g'(x)$$

Exemple 1

Find the derivatives of the following functions.

1. $f(x) = \cos(2x+1) - \sin(x^2-3)$

$$f'(x) = -(2x+1)' \sin(2x+1) - (x^2-3)' \cos(x^2-3)$$
$$= -2\sin(2x+1) - 2x\cos(x^2-3)$$

2. $h(x) = \dfrac{\sin x}{\cos^2 x}$ $\Rightarrow$

$$h'(x) = \frac{\cos(x)\cdot\cos^2(x) - \sin(x)\cdot(-2\sin(x)\cos(x))}{\cos^4(x)}$$

3. $T(\theta) = \sin(4\theta)\cos^2\theta$

$$T'(\theta) = \sin'(4\theta)\cos^2(\theta) + \sin(4\theta)(\cos^2(\theta))'$$
$$= 4\cos(4\theta)\cos^2(\theta) + \sin(4\theta)(-2\sin(\theta)\cos(\theta))$$

4. $y = \sec x$

$$y' = \left(\frac{1}{\cos(x)}\right)' = \left(\cos^{-1}(x)\right)' = (-1)\left(\cos'(x)\cdot\cos^{-2}(x)\right)$$
$$= (-1)(-\sin(x)\cos^{-2}(x)) = \sin(x)\cos^{-2}(x)$$

5. $f(x) = \csc x$

$$f'(x) = \frac{1}{\sin(x)} = \left(\sin^{-1}(x)\right)' = (-1)\left(\sin'(x)\sin^{-2}(x)\right) = (-1)\cos(x)\sin^{-2}(x)$$

6. $h(x) = \cot x$

$$h'(x) = \left(\frac{\cos(x)}{\sin(x)}\right)' = \frac{\cos'(x)\cdot\sin(x) - \cos(x)\cdot\sin'(x)}{\sin^2(x)} = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}$$
$$= \frac{-1}{\sin^2(x)}$$

7. $g(x) = \cos(e^x)\ln(\cos(e^x))$

$$g'(x) = \cos'(e^x)\ln(\cos(e^x)) + \cos(e^x)\ln'(\cos(e^x))$$
$$= -(e^x)'\sin(e^x)\ln(\cos(e^x)) + \cos(e^x)\cdot\frac{\cos'(e^x)}{\cos(e^x)}$$
$$= -e^x\sin(e^x)\ln(\cos(e^x)) + \cos(e^x)\cdot\frac{(-e^x\sin(e^x))}{\cos(e^x)}$$

Exercise 3

Consider the following function $f(x) = x\cos x$.

1. Determine the equation of the tangent line to f at $x = \pi$.

$f(\pi) = \pi \cos(\pi) = -\pi$

$y = f'(\pi)(x-\pi) + f(\pi)$

We need to find $f'(\pi)$.

$f'(x) = (x\cos(x))' = \cos(x) + x(-\sin(x)) = \cos(x) - x\sin(x)$

$\Rightarrow f'(\pi) = \cos(\pi) - \pi \cdot \sin(\pi) = -1 - \pi(0) = -1$

so that,

$y = -1 \cdot (x-\pi) + (-\pi) = x + \pi - \pi = -x$

Hence, $y = -x$

2. What is the equation of the line perpendicular to the tangent line to f at $x = \pi$.

The equation of the line $\perp$ to $y = -x$

$y = x + b$. (since $D_1 \perp D_2$ iff $m_1 \cdot m_2 = -1$)

Now, at $x = \pi$, we have $f(\pi) = -\pi$.

$\Rightarrow -\pi = \pi + b \Rightarrow b = -2\pi$

Hence, $y = -x + 2\pi$

3. Determine the rate of change of f on $[0, \pi]$

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{-\pi - 0}{\pi - 0} = -1$$

Exercise 4

Consider the following function $f(x)=\sin(\sqrt{x})$.

1. Find the derivative of f

$$f'(x) = (\sqrt{x})' \cos(\sqrt{x}) = \frac{1}{2\sqrt{x}} \cos(\sqrt{x})$$

2. Determine the equation of the tangent line to f at $x=\pi^2$.

$$\Rightarrow f(\pi^2) = \sin(\sqrt{\pi^2}) = \sin(\pi) = 0$$

$$y = f'(\pi)(x-\pi^2) + f(\pi^2)$$

From ①: $f'(\pi^2) = \frac{1}{2\sqrt{\pi^2}} \cos(\sqrt{\pi^2}) = \frac{1}{2\pi}\cos(\pi) = \frac{-1}{2\pi}$

So, $$y = -\frac{1}{2\pi}(x-\pi^2) + 0$$

$$= -\frac{1}{2\pi}x + \frac{\pi}{2}$$

3. Calculate $f''(\pi)$.

By ①, we have:

$$f'(x) = \frac{1}{2\sqrt{x}} \cos(\sqrt{x})$$

$$\Rightarrow f''(x) = \left(\frac{1}{2} x^{-1/2} \cos(\sqrt{x})\right)'$$

$$= \frac{1}{2}\left(-\frac{1}{2}\right) x^{-1/2-1} \cos(\sqrt{x}) + \frac{1}{2} x^{-1/2}\left(-(\sqrt{x})' \sin\sqrt{x}\right)$$

$$= -\frac{1}{4x^{3/2}} \cos(\sqrt{x}) + \frac{1}{2\sqrt{x}} \cdot \left(-\frac{1}{2\sqrt{x}} \sin\sqrt{x}\right)$$

$$= -\frac{1}{4x^{3/2}} \cos\sqrt{x} - \frac{1}{4x} \sin(\sqrt{x})$$

$$\Rightarrow f''(\pi^2) = -\frac{1}{4(\pi^2)^{3/2}} \cos(\sqrt{\pi^2}) - \frac{1}{4\pi^2} \sin(\sqrt{\pi^2})$$

$$= -\frac{1}{4\pi^3}\cos(\pi) - \frac{1}{4\pi^2}\sin(\pi) = \frac{-1}{4\pi^3}(-1) - \frac{1}{4\pi^2}(0)$$

$$= \frac{1}{4\pi^3}$$

Exercise 5

Find the derivatives of the following functions.

1. $f(x) = \dfrac{\sin(x)+1}{\cos(x)+3}$
2. $g(x) = \sqrt{3x^4 + e^x + \sin(x)}$
3. $h(x) = \cos\left(x^2 + 2x + 1\right)$
4. $k(x) = \sin\left(e^{-3x} + \ln(x)\right)$
5. $l(x) = \ln(2\tan(x) + 5)$

① $$f'(x) = \frac{(\sin(x)+1)'(\cos(x)+3) - (\sin(x)+1)(\cos(x)+3)'}{(\cos(x)+3)^2}$$

$$= \frac{\cos(x)(\cos(x)+3) - (\sin(x)+1)(-\sin(x))}{(\cos(x)+3)^2}$$

② $$g'(x) = \left(\left(3x^4 + e^x + \sin(x)\right)^{1/2}\right)' = \frac{1}{2}\left(3x^4+e^x+\sin(x)\right)'\left(3x^4+e^x+\sin(x)\right)^{1/2-}$$

$$= \frac{1}{2}\left(12x^3 + e^x - \sin(x)\right)\left(3x^4+e^x+\sin(x)\right)^{-1/2}$$

③ $$h'(x) = -\left(x^2+2x+1\right)' \sin\left(x^2+2x+1\right)$$

$$= -(2x+2)\sin\left(x^2+2x+1\right)$$

④ $$k'(x) = \left(e^{-3x} + \ln(x)\right)' \cos\left(e^{-3x} + \ln(x)\right)$$

$$= \left(-3e^{-3x} + \frac{1}{x}\right)\cos\left(e^{-3x} + \ln(x)\right)$$

⑤ $$l'(x) = \frac{(2\tan(x)+5)'}{2\tan(x)+5}$$

$$= \frac{2\sec^2(x)}{2\tan(x)+5}$$

3. Applications

Problem 1

The position s in (cm) of some object at time t in (second) is given by the following expression

$$s(t) = 1200\, e^{-\frac{t}{10}} \sin\left(\frac{\pi}{20}\cdot t\right)$$

1. Determine the position of this object at $t = 10$s.
2. Determine the position of this object in the long term (as $t \to \infty$).
3. Calculate the velocity of this object at $t = 10$s.
4. Deduce the velocity of this object in the long term (as $t \to \infty$).

① At $t = 10s \Rightarrow s(10) = 1200\, e^{-\frac{10}{10}} \sin\left(\frac{\pi}{20}\cdot 10\right)$

$$= 1200\, e^{-1} \cdot \sin\left(\frac{\pi}{2}\right) = 1200\, e^{-1}\,(1)$$

$$= 441.455 \text{ cm}$$

② $$-1 \le \sin\left(\frac{\pi}{20}\cdot t\right) \le 1$$

$$\Rightarrow -1200\, e^{-\frac{t}{10}} \le 1200\, e^{-\frac{t}{10}} \sin\left(\frac{\pi}{20}\cdot t\right) \le 1200\, e^{-\frac{t}{10}}$$

Using sandwich Theorem $\to 0$

③ $v(t) = s'(t)$

$$= \left(1200\, e^{-\frac{t}{10}}\right)' \sin\left(\frac{\pi}{20}t\right) + \left(1200\, e^{-\frac{t}{10}}\right)\left(\sin\left(\frac{\pi}{20}\cdot t\right)\right)'$$

$$= 1200\left(-\frac{1}{10}\right) e^{-\frac{t}{10}} \sin\left(\frac{\pi}{20}t\right) + 1200\, e^{-\frac{t}{10}}\left(\frac{\pi}{20}\right)\cos\left(\frac{\pi}{20}\cdot t\right)$$

$$= -120\, e^{-t/10} \sin\left(\frac{\pi}{20}t\right) + 60\pi\, e^{-\frac{t}{10}} \cos\left(\frac{\pi}{20}t\right)$$

at $\underline{t = 10s}$

$$v(10) = -120\, e^{-\frac{10}{10}} \sin\left(\frac{\pi}{20}\cdot 10\right) + 60\pi\, e^{-\frac{10}{10}} \cos\left(\frac{\pi}{20}\cdot 10\right)$$

$$= -120\, e^{-1}\cdot(1) + 60\pi\, e^{-1}\,(0) = -120\, e^{-1}$$

$$= 44.1455 \text{ cm/s}$$

④ Using the same idea in ② we can show that $v(t) \to 0$ as $t \to \infty$.

Problem 2

Consider the following function

$$f(x) = e^x \cos(x) \quad x \in [0, \frac{\pi}{2}].$$

Determine the absolute maximum and the absolute minimum of f.

$$f'(x) = (e^x)' \cos(x) + e^x \cos'(x)$$

$$= e^x \cos(x) + e^x(-\sin(x))$$

$$= e^x(\cos(x) - \sin(x))$$

$f'(x) = 0 \Rightarrow \cos(x) = \sin(x)$

on $[0, \frac{\pi}{2}]$, we have $\cos(x) = \sin(x)$ if $x = \frac{\pi}{4}$

and since $x = \frac{\pi}{4} \in [0, \frac{\pi}{2}] \Rightarrow x = \frac{\pi}{4}$ is a C.P for $f(x)$. Also, $x = 0$ and $x = \frac{\pi}{2}$ are both C.P for $f(x)$.

x	$f(x)$	
0	$f(0) = e^0 \cos(0) = 1.(1) = 1$	
$\frac{\pi}{4}$	$f(\pi/4) = e^{\pi/4} \cos(\frac{\pi}{4}) = 1.550$	$\leftarrow$ abs max
$\frac{\pi}{2}$	$f(\pi/2) = e^{\pi/2} \cos(\pi/2) = e^{\pi/2}.(0) = 0$	$\leftarrow$ abs min