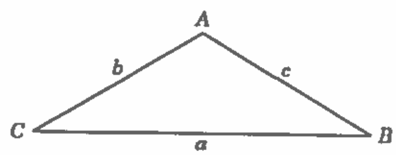
University of Ottawa
Department of math. & stat.

Geometric Vectors

1. Prerequisite Skills

Sine and cosine Laws

In a triangle, the dimensions of the sides are proportional to the sinuses of the angles to these sides.



Sine Law :

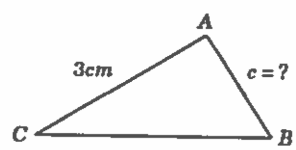
$$\boxed{\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}}$$

Cosine Law :

$$\boxed{a^2=b^2+c^2-2cb\cos A \quad , \quad b^2=a^2+c^2-2ca\cos B \quad , \quad c^2=a^2+b^2-2ab\cos C}$$

Example 1

1. Use the sine law to find the length of side c, if $\measuredangle B=65°$ et $\measuredangle C=40°$.

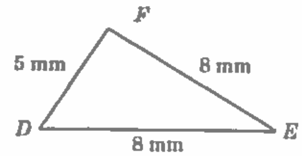


$$\frac{\sin(B)}{3}=\frac{\sin(C)}{c}$$

$$\Rightarrow c=\frac{3\sin(C)}{\sin(B)}=\frac{3\cdot\sin(40°)}{\sin(65°)}$$

$$=\frac{3\cdot(0.642)}{(0.906)}=2.127 \text{ cm}$$

2. Use the cosine law to find the measure $\angle E$



$$(DF)^2 = (FE)^2 + (DE)^2 - 2(FE)(DE)\cos(E)$$

$$5^2 = 8^2 + 8^2 - 2(8)(8)\cos(E)$$

$$\Rightarrow 25 = 128 - 128\cos(E)$$

$$\Rightarrow \cos(E) = \frac{128-25}{128} = 0.804 \Rightarrow E = \arccos(0.804) = 0.636 \text{ (rad)} = 36.48^\circ$$

2. Introduction to vectors

Definition 1

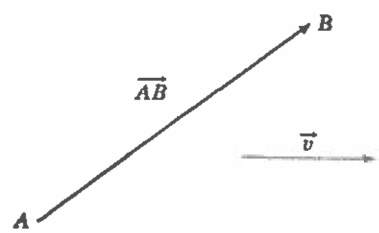
A vector is a quantity that has both magnitude and direction

- velocity
- displacement
- force
- weight
- acceleration

We can represent a vector in different ways :

i) in words, like 5 km east

ii) in a diagram as a geometric vector



iii) in a symbolic way, as $\vec{v}$ (the arrow above the letter denotes that v is a vector, not a scalar)

Note that the magnitude of vector $\overrightarrow{AB}$ or $\vec{v}$ is written as $|\overrightarrow{AB}|$ or $|\vec{v}|$.

Definition 2

A scalar is a quantity that expresses only a magnitude (with or without units). It does not specify the direction.

Example 2

Indicates whether it is a vector or scalar quantity.

1. A car travels at 50 km/h to the east.
   It is a vector quantity because it has a magnitude (50 km/h) and a direction.
2. A man has a mass of 88 kg.
   It is a scalar quantity because it has a weight of (88 kg), but it has no direction.
3. A woman skiing at a speed of 25 km/h.
   It is a scalar quantity because it has a magnitude (20 km/h), but no direction is indicated.
4. A parachutist falling at 20 km/h.
   It's a vector quantity because it has a magnitude (50 km/h) and one direction (down).

Definition 3

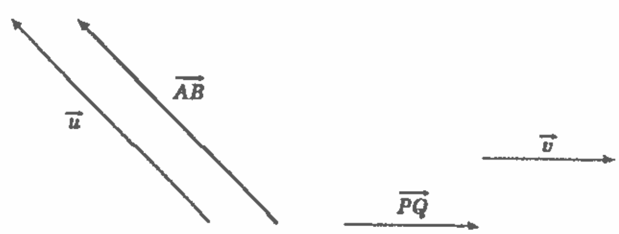
Two vectors are said to be parallel if they have the same or opposite directions (but not necessarily the same magnitude).

We may write

$$\vec{v} \parallel \overrightarrow{PQ}$$

Definition 4

Two vectors that have the same direction and magnitude are said to be equivalent or equal (their actual locations in space do not matter).

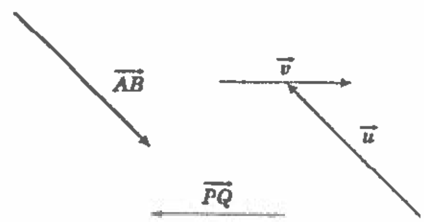


We may write

$$\vec{u} = \overrightarrow{AB}, \quad \text{and} \quad \vec{v} = \overrightarrow{PQ}$$

Definition 5

Two parallel vectors of the same length but in opposite directions are said opposite,



We may write

$$-\overrightarrow{u} = \overrightarrow{AB}, \qquad \text{and} \qquad \overrightarrow{v} = -\overrightarrow{PQ}$$

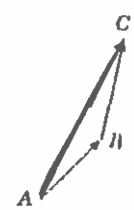
2.1. Addition and Substraction of Vectors

Definition 6

Let two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$. The vector $\overrightarrow{w}$ is the geometric sum of $\overrightarrow{u}$ and $\overrightarrow{v}$ and we write

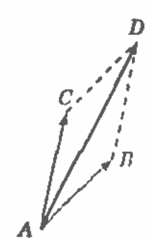
$$\overrightarrow{w} = \overrightarrow{u} + \overrightarrow{v}$$

we can add vectors by joining them head to tail.



$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

Another way to get the geometric sum of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as shown in the following figure is to complete the parallelogram and trace the vector $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$ (the diagonal).

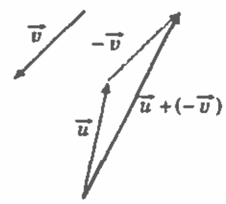


Since, $\overrightarrow{BD} = \overrightarrow{AC}$ and $\overrightarrow{AB} = \overrightarrow{CD}$, we have $\quad \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AB} + \overrightarrow{BD}$

Remark

Using the vector addition operation and the concept of opposing vectors, the vector subtraction will be

$$\vec{w} = \vec{u} + (-\vec{v})$$



Thus, the addition of two **opposing vectors is the null vector.** For example $\vec{u} - \vec{u} = \vec{0}$. This vector has no direction.

Properties of the addition of vectors

Let the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ and let the scalars $c, d \in \mathbb{R}$. So,

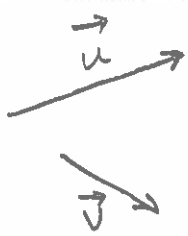
1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ **(commutativity)**
2. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ **(associativity)**
3. There is an **additive neutral element** $\vec{0}$ (null vector.) such that $\vec{u} + \vec{0} = \vec{u}$

Example 3

Simplify the expression. $[(\vec{p} + \vec{q}) - \vec{p}] - \vec{q} =$ $\vec{p} + \vec{q} - \vec{p} - \vec{q} = \vec{p} - \vec{p} + \vec{q} - \vec{q} = \vec{0}$

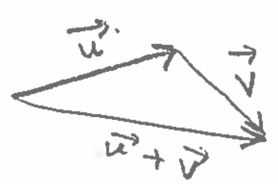
Example 4

Let the vectors $\vec{u}$, $\vec{v}$

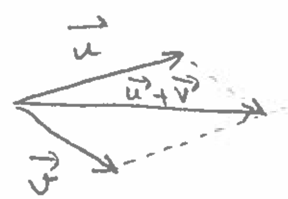


1. Find $\vec{u} + \vec{v}$
2. Find $\vec{u} - \vec{v}$

1)



or



2)

2.2. Multiplying a Vector by a Scalar

2.2.1 Multiply a vector by a scalar k

Definition 7

If we take a vector $\overrightarrow{v}$ and multiply it by a scalar k (any real number), we are performing scalar multiplication and have produced the scalar multiple $k\overrightarrow{v}$.

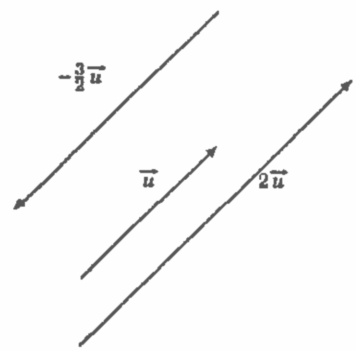
Note that, the vector $k\overrightarrow{v}$ will be $|k|$ times as long as $\overrightarrow{v}$ and it will be parallel to $\overrightarrow{v}$.

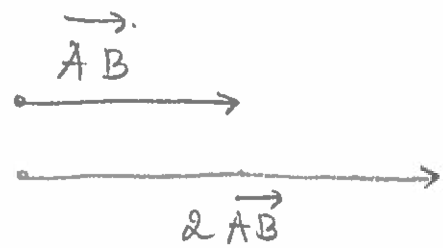
The magnitude of $k\overrightarrow{v}$ is $|k\overrightarrow{v}| = |k||\overrightarrow{v}|$

1. If $k > 0$, $k\overrightarrow{v}$ is in the same direction as $\overrightarrow{v}$.
2. If $k < 0$, $k\overrightarrow{v}$ is in the opposite direction to $\overrightarrow{v}$.
3. If $|k| > 1$, $k\overrightarrow{v}$ is longer than $\overrightarrow{v}$.
4. If $|k| < 1$, $k\overrightarrow{v}$ is shorter than $\overrightarrow{v}$.

Example 5

Consider the vector $\overrightarrow{u}$ below.





$|2\overrightarrow{AB}| = |2|\,|\overrightarrow{AB}|$
$= 2\,|\overrightarrow{AB}|$

The vector $-\frac{3}{2}\overrightarrow{u}$ is a scalar multiple of $\overrightarrow{u}$. Its length is therefore $|-\frac{3}{2}||\overrightarrow{u}| = \frac{3}{2}|\overrightarrow{u}|$.

2.2.2 Collinear vectors

Definition 8

Let two non-zero vectors $\overrightarrow{u}$ and $\overrightarrow{v}$. We say that $\overrightarrow{u}$ and $\overrightarrow{v}$ are **collinear** if there is a number $k \in \mathbf{R}$ such that $\overrightarrow{u} = k\overrightarrow{v}$.

In other words, two vectors are collinear if one is a multiple of the other.

Example 6

The vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ are collinear because $\overrightarrow{u} = -2\overrightarrow{v}$.

Remark

- The definition above ensures that the actual number k is different from 0.
- The null vector is collinear with all the vectors.
- Two collinear vectors are parallel.

2.2.3 Product properties of a vector by a scalar

Let the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ and the scalars $c, d \in \mathbb{R}$. So,

1. $c\cdot(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$ **(distributivity)**
2. $c\cdot(d\vec{u}) = (cd)\vec{u}$ **(associativity)**
3. There is a **multiplicative neutral element** 1 such that $1\vec{u} = \vec{u}$

Example 7

Simplify vector expression $2(\vec{u}-\vec{v})+3(\vec{u}+\vec{v})$.

$$2(\vec{u}-\vec{v})+3(\vec{u}+\vec{v}) = 2\vec{u}-2\vec{v}+3\vec{u}+3\vec{v}$$
$$= 2\vec{u}+3\vec{u}-2\vec{v}+3\vec{v}$$
$$= 5\vec{u}-\vec{v}$$

2.2.4 Linear combinations of vectors

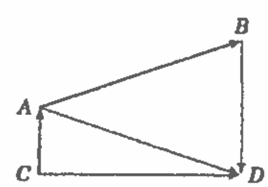
Let the vectors $\vec{u}$, $\vec{v}$ and the scalars $c, d \in \mathbb{R}$. The following quantity

$$\boxed{c\vec{u}+d\vec{v}}$$

is called a **linear combination** of vectors $\vec{u}$ and $\vec{v}$.

Exercise 1

Let $\vec{u} = \overrightarrow{CA}$ and $\vec{v} = \overrightarrow{AB}$. Express $\overrightarrow{BD}$, $\overrightarrow{AD}$ and $\overrightarrow{CD}$ as linear combinations of $\vec{u}$ and $\vec{v}$



Here $\overrightarrow{BD} = 2\overrightarrow{AC}$
$= -2\overrightarrow{CA}$
$= -2\vec{u}$

$\overrightarrow{BD} = -2\overrightarrow{CA} = -2\vec{u}$	$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$ $= \vec{v} - 2\vec{u}$	$\overrightarrow{CD} = \overrightarrow{CA} + \overrightarrow{AD}$ $= \vec{u} + \vec{v} - 2\vec{u}$ $= \vec{v} - \vec{u}$

Remark

Chasles relation

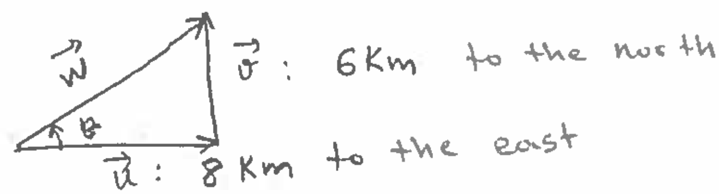
Whatever the relative positions of points A, B and C, we have

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{0}$$

2.3. Applications of Vector Addition

Problem 1

1. Draw the resultant of the pair of orthogonal components. Then determine the magnitude and direction of the resultant.



$\vec{w} = \vec{u} + \vec{v}$. The magnitude of $\vec{w}$ is

$|\vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2 \Rightarrow |\vec{w}| = \sqrt{|\vec{u}|^2 + |\vec{v}|^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$

(Here, we used the Pythagorean theorem)

$\sin(\theta) = \frac{6}{10} \Rightarrow \theta = \sin^{-1}\left(\frac{6}{10}\right) = 36.87^\circ$

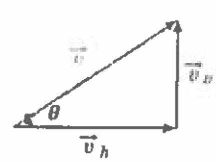
2.4. Decomposition of vectors into rectangular components

We can decompose a given vector into two perpendicular vectors whose sum is the given vector.

Definition 9

Let a vector $\vec{v}$ at an angle θ with respect to the horizontal. The rectangular components of $\vec{v}$ are the vectors $\vec{v}_h$ and $\vec{v}_v$ such as,

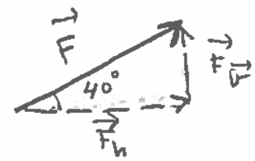
1. $\vec{v} = \vec{v}_h + \vec{v}_v$
2. $\vec{v}_h$ et $\vec{v}_v$ are perpendicular



Example 8

A tow truck is pulling a car from a ditch. The tension in the cable is $|\vec{F}| = 15000$ N at an angle of 40° to the horizontal.

1. Draw a diagram that shows the horizontal and vertical of $\vec{F}$.



$\theta = 40°$

2. Determine the magnitude of the horizontal and vertical components.

$$\sin(\theta) = \frac{|\vec{F_v}|}{|\vec{F}|} \Rightarrow |\vec{F_v}| = \sin(\theta)|\vec{F}| = \sin(40°)\,15000$$
$$= 9641.814 \text{ N}$$

$$\cos(\theta) = \frac{|\vec{F_h}|}{|\vec{F}|} \Rightarrow |\vec{F_h}| = \cos(\theta)|\vec{F}| = \cos(40°)\,15000$$
$$= 11490.67 \text{ N}$$

Remark

1. Given the magnitude of $\vec{v}$, the magnitude of the horizontal component and the magnitude of the vertical component are calculated by

$$\boxed{|\vec{v}_h| = |\vec{v}|\cos\theta \quad \text{et} \quad |\vec{v}_v| = |\vec{v}|\sin\theta}$$

2. Using the Pythogorean Theorem $|\vec{v}| = \sqrt{|\vec{v}_h|^2 + |\vec{v}_v|^2}$