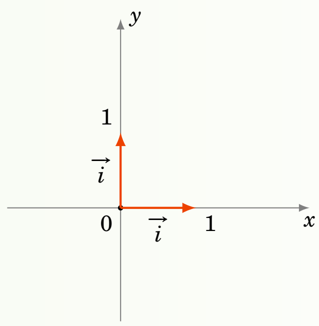
University of Ottawa
Department of math. & stat.

Dot Product

1. Cartesian Vectors

Definition 1

In a Cartesian plane, the **unit** vectors are $\vec{i} = [1,0]$ and $\vec{j} = [0,1]$.



Remark

- A Cartesian plane is denoted by $\left(O, \vec{i}, \vec{j}\right)$.
- The magnitude of $\vec{i}$ is equal to 1.
- The magnitude of $\vec{j}$ is equal to 1.

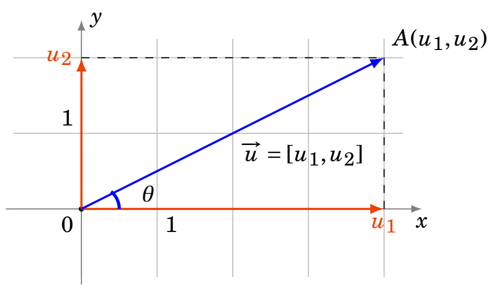
Definition 2

A **Cartesian vector** $\vec{u}$ in the Cartesian plane $\left(O, \vec{i}, \vec{j}\right)$ is defined by

$$\vec{u} = [u_1, u_2] \qquad \text{where} \qquad u_1, u_2 \in \mathbb{R}$$

Note that $\vec{u} = [u_1, u_2]$ is also called position vector.

Remark



- u_1 is the horizontal component and u_2 is the vertical component of the vector $\vec{u}$.
- Since $u_1 = |\vec{u}|\cos\theta$ and $u_2 = |\vec{u}|\sin\theta$, we write

$$\boxed{\vec{u} = \left[|\vec{u}|\cos\theta, |\vec{u}|\sin\theta\right]}$$

The linear combinations of cartesian vectors allow us to see if it is possible, to characterize a vector by its components.

Definition 3

Let $\vec{u}$ be a vector of the Cartesian plane. So, there is a unique couple (a,b) of the plane such that :

$$\vec{u} = a\vec{i} + b\vec{j}$$

Example

if $\vec{a} = 3\vec{i} - 5\vec{j}$ and $\vec{b} = 2\vec{i} + 7\vec{j}$, express $5\vec{a} + 2\vec{b}$ as a linear combination of $\vec{i}$ and $\vec{j}$.

$$\begin{aligned} 5\vec{a} + 2\vec{b} &= 5(3\vec{i} - 5\vec{j}) + 2(2\vec{i} + 7\vec{j}) \\ &= 15\vec{i} - 25\vec{j} + 4\vec{i} + 14\vec{j} \\ &= 19\vec{i} - 11\vec{j} \end{aligned}$$

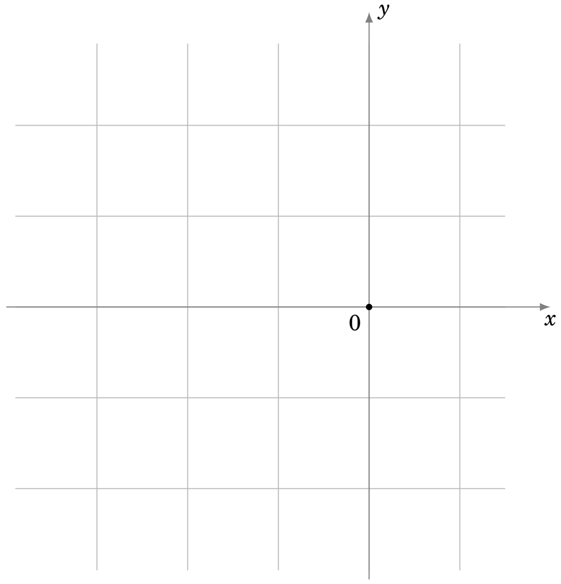
Definition 4

The **magnitude** of the vector $\vec{u} = [u_1, u_2]$ is given by

$$|\vec{u}| = \sqrt{u_1^2 + u_2^2}$$

Exercise 1

Draw the vectors positions of $\vec{u} = [-3,2]$, $\vec{v} = [0,-2]$ and $\vec{w} = \left[\frac{4}{5}, \frac{3}{5}\right]$ and calculate their magnitude.



$$
\begin{aligned}
|\vec{u}| &= \sqrt{(-3)^2+2^2}=\sqrt{13} \\
|\vec{v}| &= \sqrt{0^2+(-2)^2}=2 \\
|\vec{w}| &= \sqrt{\left(\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16+9}{25}}=1
\end{aligned}
$$

1.1. Adding, Subtracting Vectors and Multiplying a Vector by a Scalar

Proposition 1

Let two vectors $\vec{u}=[u_1,u_2]$ and $\vec{v}=[v_1,v_2]$ and let k be a real number.

The cartesian vector $\vec{u}+\vec{v}$ is the sum of $\vec{u}$ and $\vec{v}$ and we write

$$\boxed{\vec{u}+\vec{v}=[u_1+v_1\,,\,u_2+v_2]}$$

The cartesian vector $\vec{u}-\vec{v}$ is the difference of $\vec{u}$ and $\vec{v}$ and we write

$$\boxed{\vec{u}-\vec{v}=[u_1-v_1\,,\,u_2-v_2]}$$

The cartesian vector $k\vec{v}$ is the scalar multiple of $\vec{v}$ and we write

$$\boxed{k\vec{v}=[kv_1\,,\,kv_2]}$$

To add two vectors, we add component by component. Likewise for $\vec{u}-\vec{v}$. For multiplication by a scalar, just multiply each component by the scalar.

Exercise 2

Consider tow vectors $\vec{u}=[-2,1]$ and $\vec{v}=[5,3]$. Find

1. $\vec{u}+\vec{v}$

$$
\begin{aligned}
\vec{u}+\vec{v} &= [-2,1]+[5,3] \\
&= [-2+5,1+3]=[3,4]
\end{aligned}
$$

2. $|\vec{u}-\vec{v}|$

$$
\begin{aligned}
|\vec{u}-\vec{v}| &= \big|[-2,1]-[5,3]\big| \\
&= \big|[-2-5,1-3]\big|=\big|[-7,-2]\big| \\
&= \sqrt{(-7)^2+(-2)^2}=\sqrt{53}
\end{aligned}
$$

3. $3\overrightarrow{u}+5\overrightarrow{v}$

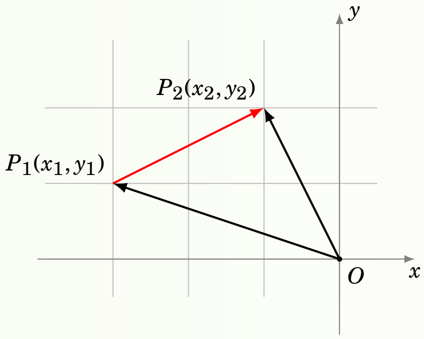
$$
\begin{aligned}
3\overrightarrow{u}+5\overrightarrow{v} &= [3(-2),3(1)]+[5(5),5(3)]\\
&= [-6,3]+[25,15]\\
&= [-6+25,3+15]\\
&= [19,18]
\end{aligned}
$$

1.2. Cartesian Vector Between Two Points

The following proposition is frequently used.

Proposition 2

Consider the following points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$



then the vector $\overrightarrow{P_1P_2}$ is

$$\boxed{\overrightarrow{P_1P_2}=[x_2-x_1,y_2-y_1]}$$

with magnitude

$$\boxed{\left|\overrightarrow{P_1P_2}\right|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$$

Example

Find the components and the magnitude of $\overrightarrow{PQ}$, where $P(1,2)$ and $Q(0,-1)$.

$$
\begin{aligned}
\overrightarrow{PQ} &= [0-1,-1-2]\\
&= [-1,-3]
\end{aligned}
$$

$$
\begin{aligned}
\left|\overrightarrow{PQ}\right| &= \sqrt{(-1)^2+(-3)^2}\\
&= \sqrt{10}.
\end{aligned}
$$

Two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ can be multiplied in two ways, according to the **Dot product** whose result is a **scalar** and according to **the Cross product** whose result is a **vector**.

2. Dot product

The Dot Product is a very important concept in the study of vectors because of its many applications. One can use it to determine the work done, the angle formed by two vectors and the projection of one vector on another.

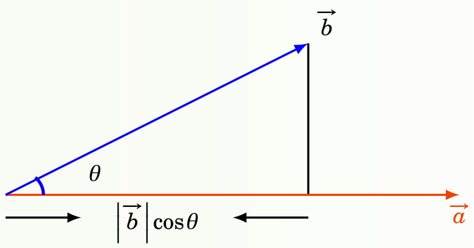
2.1. Geometric approach

Definition 5

Let $\overrightarrow{a}$ and $\overrightarrow{b}$ two vectors. The **Dot product** $\overrightarrow{a}\cdot\overrightarrow{b}$ is defined by

$$\overrightarrow{a}\cdot\overrightarrow{b}=\left|\overrightarrow{a}\right|\cdot\left|\overrightarrow{b}\right|\cos\theta$$

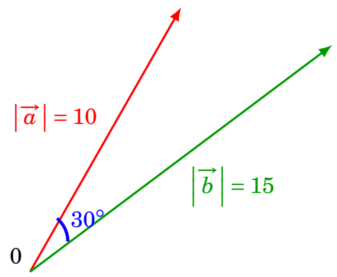
where $0°\leqslant\theta\leqslant 180°$ is the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ when their origins coincide.



It is important to note that the Dot Product is a real number and not a vector.

Example 1

Let $\overrightarrow{a}$ be a vector with magnitude 10 and let $\overrightarrow{b}$ be a vector with magnitude 15. calculate $\overrightarrow{a}\cdot\overrightarrow{b}$.



$$\begin{aligned}\overrightarrow{a}\cdot\overrightarrow{b} &= \left|\overrightarrow{a}\right|\cdot\left|\overrightarrow{b}\right|\cos 30° \\ &= 10\cdot 15\cdot\frac{\sqrt{3}}{2}\simeq 129.9\end{aligned}$$

2.2. Algebraic approach

Theorem 1

A Dot Product of $\vec{a} = [a_1, a_2]$ and $\vec{b} = [b_1, b_2]$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

Example 2

Calculate $\vec{a} \cdot \vec{b}$ if $\vec{a} = 2\vec{i} - 3\vec{j}$ and $\vec{b} = 5\vec{i} + 3\vec{j}$

$\vec{a} \cdot \vec{b} = (2)(5) + (-3)(3) = 10 - 9 = 1$

Dot Product properties

Consider the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $k \in \mathbb{R}$.

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (**commutativity**)
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ (**distributivity**)
4. $(k \cdot \vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k \cdot \vec{b})$ (**associativity**)
5. if $\vec{a}$ and $\vec{b}$ are non-zero vectors, then these vectors are **orthogonal** (or **perpendicular**) if and only if $\vec{a} \cdot \vec{b} = 0$. In other words,
   (a) if $\vec{a}$ and $\vec{b}$ are **orthogonal**, then $\vec{a} \cdot \vec{b} = 0$
   (b) if $\vec{a} \cdot \vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are **orthogonal**.

Exercise 3

Use Dot Product properties to simplify each expression

1. $(k\vec{u}) \cdot (\vec{u} + \vec{v}) =$

2. $(\vec{r} + \vec{s}) \cdot (\vec{r} - \vec{s})$

3. Applications of the Dot Product

3.1. Work Done

Definition 6

The **work done by a constant force** $\vec{f}$ whose point application moves according to the displacement $\vec{s}$ is given by

$$\vec{W} = \vec{f} \cdot \vec{s}$$

Example 3

Angela has entered the wheelchair division of a marathon race. While training, she races her wheelchair up a $s = 300$ m with a constant force of 500 N applied at an angle of $\theta = 30°$ to the surface of the hill. Determine the work done by Angela.

The work done W is

$$W = \vec{f} \cdot \vec{s} = \left|\vec{f}\right| \left|\vec{s}\right| \cos\theta$$
$$= (500)(300)\cos(30°) \approx 129\,904$$

The work done by Angela is approximately 129 900 $N \cdot m$ or 129 900 J.

3.2. Angle Between two Vectors

Theorem 2

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors, the angle θ, $0° \leqslant \theta \leqslant 180°$, between the vectors $\vec{a}$ and $\vec{b}$ is given by the formula :

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\left|\vec{a}\right| \left|\vec{b}\right|}$$

Remark

- Two vectors $\vec{a}$ and $\vec{b}$ are **orthogonal** (or **perpendicular**) iff the angle between them is $\pi/2$, and consequently $\vec{a} \cdot \vec{b} = 0$.
- Two vectors $\vec{a}$ and $\vec{b}$ are **collinear** (or **parallel**) iff the angle between them is 0 or π.

  since, if $\vec{b} = k\vec{a}$, then

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\left|\vec{a}\right| \left|\vec{b}\right|} = \frac{\vec{a} \cdot k\vec{a}}{|k| \left|\vec{a}\right| \left|\vec{a}\right|} = \frac{k}{|k|}$$

  1. if $k > 0$, $\cos\theta = 1 \implies \theta = 0$
  2. if $k < 0$, $\cos\theta = -1 \implies \theta = \pi$

Exercise 4

Find the angle between $\vec{a} = [-1.5, 3]$ and $\vec{b} = [2, 1]$

$$\begin{aligned}
\cos\theta &= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\left|\vec{b}\right|} \\
&= \frac{(-1.5)(2)+(3)(1)}{\sqrt{(-1.5)^2+3^2}\sqrt{2^2+1^2}} \\
&= \frac{0}{\sqrt{11.25}\sqrt{5}} \\
&= 0
\end{aligned}$$

So, the vectors are orthogonal.

Exercise 5

Suppose that $\vec{a}$ and $\vec{b}$ are two non-zero vectors. Prove that $|\vec{a}\cdot\vec{b}| \leqslant |\vec{a}|\left|\vec{b}\right|$

$$\begin{aligned}
|\cos\theta| &\leqslant 1 \\
|\vec{a}|\left|\vec{b}\right||\cos\theta| &\leqslant |\vec{a}|\left|\vec{b}\right| \\
|\vec{a}\cdot\vec{b}| &\leqslant |\vec{a}|\left|\vec{b}\right|
\end{aligned}$$