1. Cross Product

We introduce the cross product of two vectors $\vec{a}$ and $\vec{b}$ of dimension three, denoted by $\vec{a} \times \vec{b}$.

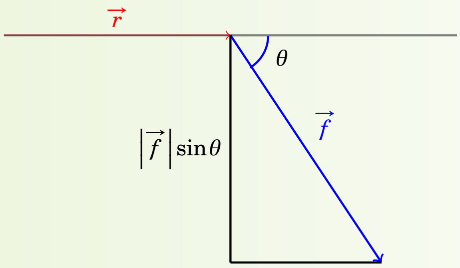
Note that the dot product $\vec{a} \cdot \vec{b}$ is a scalar. However, the cross product $\vec{a} \times \vec{b}$ is a vector.

Mise-en-situation

The following figure shows the force $\vec{f}$ when we screw a bolt. The effect produced is called moment $\vec{\tau}$ (tau) which depends on two factors :

1. $|\vec{r}|$, the distance between the center of the bolt and the application point of the force $\vec{f}$.
2. $\left(\left|\vec{f}\right| \sin\theta\right)$, the magnitude of the orthogonal force at the wrench

where θ is the angle between the vectors $\vec{r}$ and $\vec{f}$.



The magnitude of the moment vector is defined by

$$|\vec{\tau}| = |\vec{r}|\left|\vec{f}\right| \sin\theta$$

The unit vector $\hat{n}$ illustrates the direction of a bolt produced by its rotation. Hence the definition of the **moment**,

$$\vec{\tau} = \left(|\vec{r}|\left|\vec{f}\right| \sin\theta\right)\hat{n}$$

If we write $\vec{\tau} = \vec{r} \times \vec{f}$, and we call it **cross product**, we will have the following definition.

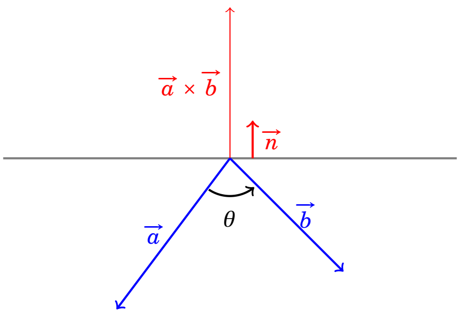
Definition 1

Given two vectors $\vec{a}$ and $\vec{b}$ of dimensions 3, the **cross product** of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is a new vector defined as

$$\vec{a} \times \vec{b} = \left(|\vec{a}|\left|\vec{b}\right| \sin\theta\right)\hat{n}$$

where $0 \leqslant \theta \leqslant \pi$ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is a vector perpendicular to both vectors $\vec{a}$ and $\vec{b}$ and its direction is given by the **rule of the right hand**

Right-hand rule



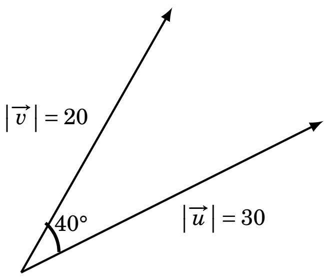
When the fingers of the right-hand turn at an angle θ from $\vec{a}$ to until $\vec{b}$, then the thumb indicates the direction of $\hat{n}$.

Remark

- if $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$.
- $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$.
- Two non-zero vectors $\vec{a}$ and $\vec{b}$ are **parallel** if and only if $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = \vec{\mathbf{0}}$.

Example

1. Suppose that $|\vec{u}| = 30$, $|\vec{v}| = 20$ and $\theta = 40°$ is the angle formed by $\vec{u}$ and $\vec{v}$.



Determine the following products.

(a) $\vec{u} \times \vec{v}$

According to the right-hand rule , the direction of $\vec{u}$ and $\vec{v}$ is outward from the page. Let $\hat{n}$ be the perpendicular unit vector to both $\vec{u}$ and $\vec{v}$ with direction outward from the page. then,

$$\begin{aligned}\vec{u} \times \vec{v} &= \left(|\vec{u}||\vec{v}|\sin\theta\right)\hat{n}\\ &= \left((30)(20)\sin(40°)\right)\hat{n}\\ &\approx 38,6\,\hat{n}\end{aligned}$$

(b) $\vec{v} \times \vec{u}$

Using the right-hand rule, the direction of $\vec{v} \times \vec{u}$ is into the page. Thus,

$$\vec{v} \times \vec{u} = \left((20)(30)\sin(40°)\right)(-\hat{n}) \approx -38,6\,\hat{n}$$

We note that the vector product is not commutative, that is to say

$$\overrightarrow{u} \times \overrightarrow{v} \quad \neq \quad \overrightarrow{v} \times \overrightarrow{u}$$

2. Calculate the cross products $\overrightarrow{i} \times \overrightarrow{j}$ and $\overrightarrow{j} \times \overrightarrow{i}$.

   The vectors $\overrightarrow{i}$ and $\overrightarrow{j}$ are the bases of an orthogonal coordinate system. So, they are perpendicular and with magnitude of 1 each. By the rule of the right hand, from $\overrightarrow{i}$ to $\overrightarrow{j}$, we find that the unit vector $\widehat{n} = \overrightarrow{k}$.
   So,

$$\overrightarrow{i} \times \overrightarrow{j} = \left(\left|\overrightarrow{i}\right|\left|\overrightarrow{j}\right|\sin(90°)\right)\widehat{n} = \overrightarrow{k}$$

   On the other hand, the rule of the right hand, from $\overrightarrow{j}$ to $\overrightarrow{i}$, tells us that $\widehat{n} = -\overrightarrow{k}$.
   Therefore,

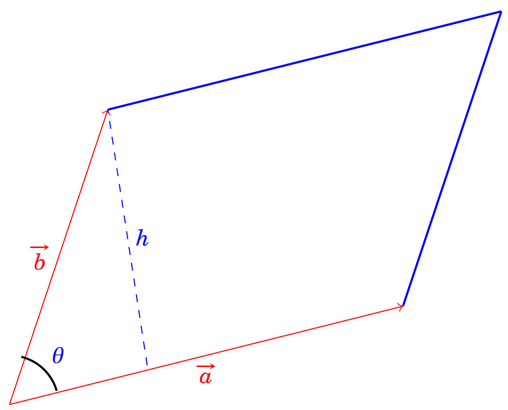
$$\overrightarrow{j} \times \overrightarrow{i} = -\overrightarrow{k}$$

Note that, the order of the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ is very important to determine the direction of their cross product.

The magnitude of $\overrightarrow{u} \times \overrightarrow{v}$ can be determined by examining the parallelogram formed by $\overrightarrow{u}$ and $\overrightarrow{v}$.

Example : Geometric interpretation of $\overrightarrow{a} \times \overrightarrow{b}$

Show that the area of the parallelogram formed by the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ is equal to the magnitude of the cross product $\overrightarrow{a} \times \overrightarrow{b}$.



Given that $\sin\theta = \frac{h}{\left|\vec{b}\right|}$, so, $h = \left|\vec{b}\right| \sin\theta$. We find that

$$\begin{aligned}\text{Area} &= \text{base} \times \text{height} \\ &= \left|\vec{a}\right| \cdot h \\ &= \left|\vec{a}\right| \left|\vec{b}\right| \sin\theta \\ &= \left|\vec{a} \times \vec{b}\right|\end{aligned}$$