MAT 1341A . F15 Final Exam Solutions

1. For a **homogeneous** system of 1500 equations in 2015 unknowns, answer the following three questions:

- o Can the system be inconsistent? NO
- o Can the system have infinitely many solutions? YES
- o Can the system have exactly one solution? NO, since rank $A \leq 1500$, # parameters $\geq 2015-1500 > 0$.

2015 1500 A 0

A. No, Yes, No.
B. Yes, Yes, Yes.
C. Yes, Yes, No.
D. No, No, No.
E. Yes, No, Yes.
F. No, No, Yes.

2. Suppose $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Which one of the following statements is true ?

A. A^{-1} does not exist.
B. The third row vector of A^{-1} is $(-1, 1, 1)$.
C. The second row vector of A^{-1} is $(0, 1, 1)$.
D. The first row vector of A^{-1} is $(1, -1, 2)$.
E. The second column vector of A^{-1} is $(0, 2, -1)$.
F. The first column vector of A^{-1} is $(1, 2, -1)$.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & & * & \\ 0 & -1 & 0 & & & \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

3. Let A be a 13×8 matrix such that $\operatorname{rank} A = 6$. As usual, $\ker A = \{x \in \mathbf{R}^8 \mid Ax = 0\}$ and $\operatorname{col} A = \{Ax \mid x \in \mathbf{R}^8\}$. Which of the following statements is true?

A. $\dim \operatorname{col} A = 6$, and $\dim \ker A = 1$

B. $\operatorname{col} A = \mathbf{R}^{13}$, and $\dim \ker A = 6$

C. $\operatorname{col} A = \mathbf{R}^8$, and $\dim \ker A = 5$

D. $\dim \operatorname{col} A = 6$, and $\dim \ker A = 2$

E. $\dim \operatorname{col} A = 6$, and $\dim \ker A = 7$

F. $\operatorname{col} A = \{0\}$, and $\dim \ker A = 7$

rank $A = 6 \Leftrightarrow$

dim col $A = 6$

rank $A = 6$

$\Rightarrow$ dim ker $A = 8 - 6 = 2$

4. Let $X = \{(x, y, z) \in \mathbf{R}^3 \mid x + y = 0\}$. Which one of the following statements is true?

A. X is a subspace of $\mathbf{R}^3$ and $\dim X = 3$. ×

B. X is a subspace of $\mathbf{R}^2$ and $\dim X = 1$. ×

C. X is a plane in $\mathbf{R}^3$ through the origin which is parallel to the z–axis. ✓

D. X is a not a subspace of $\mathbf{R}^3$. ×

E. X is a line through the origin in $\mathbf{R}^3$. ×

F. X is a plane in $\mathbf{R}^3$ through the origin which is parallel to the x–axis. ×

5. Recall that A^t denotes the transpose of the matrix A. The dimension of

$$S = \{A \in \mathbf{M}_{33}(\mathbf{R}) \mid A = -A^t\}$$

is:

A. 1
B. 2
C. 3
D. 4
E. 6
F. 9

$$A \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = -\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & j \end{bmatrix} \Leftrightarrow A = \begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{bmatrix}$$

$$\therefore S = \operatorname{span}\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$

and the spanning set is LI. $\therefore$

$\dim S = 3$

6. Which two of the following statements are true?

I. $\{2\sin^2 x, 3\cos^2 x, 2\}$ is linearly independent in $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$. F
II. $\mathbf{R}^2$ has a spanning set with 3 vectors. ✓
III. The set of all solutions to any homogeneous linear system is a subspace. ✓
IV. If u and v are linearly independent vectors in $\mathbf{R}^2$, then $\{u, v\} = \operatorname{span}\{u, v\}$. F

A. *I* and *II*.
B. *I* and *III*.
C. *I* and *IV*.
D. *II* and *III*.
E. *II* and *IV*.
F. *III* and *IV*.

I: $2 = 2\sin^2 x + \frac{2}{3} \cdot 3\cos^2 x, \ \forall x \ \therefore$
$\{2\sin^2 x, 3\cos^2 x, 2\}$ is LD.

IV If $\{u, v\}$ is LI, then $u \neq 0, v \neq 0$,
so span$\{u, v\}$ contains
infinitely many vectors (besides
u and v.)

7. If $C = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ and D is a $3 \times n$ matrix , then the first row of the matrix CD is always

A. undefined unless $n = 2$.

B. twice the first row of D.

C. the same as the first row of D.

D. the same as the second row of D.

E. the sum of the first and the second rows of D.

F. the sum of three times the second row of D and the third row of D.

$$CD = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 3r_2 + r_3 \\ r_1 + 2r_2 + r_3 \end{bmatrix}$$

r_i = ith row of D

8. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$, find

$$\begin{vmatrix} b+4c & e+4f & h+4i \\ -3c & -3f & -3i \\ 5a & 5d & 5g \end{vmatrix} = -15. \begin{vmatrix} b+4c & e+4f & h+4i \\ c & f & i \\ a & d & g \end{vmatrix}$$

A. -120
B. -30
C. -24
D. 24
E. 30
F. 120

$$= -15 \begin{vmatrix} b & e & h \\ c & f & i \\ a & d & g \end{vmatrix} = 15 \begin{vmatrix} a & d & g \\ c & f & i \\ b & e & h \end{vmatrix}$$

$$= -15 \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = -15 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -15 . 2 = -30$$

9. The vectors $u_1 = (1,1,2)$, $u_2 = (1,-1,0)$, and $u_3 = (1,1,-1)$ form an orthogonal basis of $\mathbf{R}^3$. If we write $(1,-1,1) = a_1u_1 + a_2u_2 + a_3u_3$, what is a_2?

A. -1
B. 1
C. $-\frac{\sqrt{2}}{2}$
D. $\frac{\sqrt{2}}{2}$
E. $-\frac{1}{3}$
F. $\frac{1}{3}$

$$a_2 = \frac{(1,-1,1)\cdot u_2}{\|u_2\|^2}$$

$$= \frac{2}{2} = 1.$$

10. Let A be a 3 by 3 matrix with real entries. Which of the following statements is **equivalent to**

"A is not diagonalizable over the reals." $(*)$

A. Not all of the eigenvalues of A are real.
B. A does not have any real eigenvectors.
C. A does not have three distinct real eigenvalues.
D. A does not have three independent eigenvectors in $\mathbf{R}^3$.
E. A is upper triangular.
F. A is not invertible.

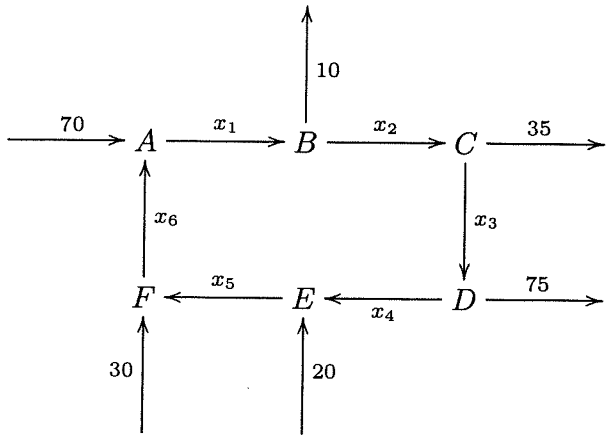
A - a matrix can fail to be diag'ble even if all evals are real. eg $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is not diag'ble (check!)

B. This statement implies $(*)$ but is not implied by $(*)$ ∵ a matrix can fail to be diag'ble even if all evecs are real - see the example for A

C. A can be diag'ble even if it has just 1 eval. For example $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is diag'ble, but has just one eval.

E. See the example in "A"
F "

11. Consider the network of streets with intersections A, B, C, D, E and F below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D, E and F during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear **equations** which describes the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \dots, 6$. (Do not perform any operations on your equations: this is done for you in (b), and *do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

Intersection	Flow in	=	Flow out
A	$x_6 + 70$	=	x_1
B	x_1	=	$10 + x_2$
C	x_2	=	$35 + x_3$
D	x_3	=	$75 + x_4$
E	$x_4 + 20$	=	x_5
F	$x_5 + 30$	=	x_6.

Constraints: $x_i \geq 0$, $i = 1, \dots, 6$ (One way streets)

$x_i \in \mathbb{Z}$, " ("# of cars")

11. b) The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\begin{array}{c} \qquad\qquad\qquad\qquad\quad s \\ \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 70 \\ 0 & 1 & 0 & 0 & 0 & -1 & 60 \\ 0 & 0 & 1 & 0 & 0 & -1 & 25 \\ 0 & 0 & 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 0 & 0 & 1 & -1 & -30 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$$\begin{aligned} x_1 &= 70 + s \\ x_2 &= 60 + s \\ x_3 &= 25 + s \\ x_4 &= -50 + s \\ x_5 &= -30 + s \\ x_6 &= s \end{aligned} \quad ; s \in \mathbb{R}$$

11. c) Find the minimum flow along BC, **using your results from (b).**
(You must justify all your answers.)

Flow along BC is x_2.

$$\left.\begin{aligned} x_1 \geq 0 &\Leftrightarrow s \geq -70 \\ x_2 \geq 0 &\Leftrightarrow s \geq -60 \\ x_3 \geq 0 &\Leftrightarrow s \geq -25 \\ x_4 \geq 0 &\Leftrightarrow s \geq 50 \\ x_5 \geq 0 &\Leftrightarrow s \geq 30 \\ x_6 \geq 0 &\Leftrightarrow s \geq 0 \end{aligned}\right\} \Rightarrow s \geq 50$$

$\therefore \; x_2 = 60 + s \geq 110.$

12. Let $W = \{(x, y, z, u) \in \mathbf{R}^4 \mid x + y - u = 0\}$.

a) <u>*Without referring to the Subspace Test*</u> briefly explain why W is a subspace of $\mathbf{R}^4$. (Remember, I am not a high dimensional being, so geometric arguments in $\mathbf{R}^4$ won't convince me.)

b) Find a basis for W.

c) Use the Gram-Schmidt algorithm to find an orthogonal basis for W.

d) Find the best approximation by a vector in W to the vector $(1, 1, 1, 0)$.

(Remember: you must justify your answers.)

a) $W = \ker \begin{array}{c} \begin{matrix} & r & s & t \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & -1 \end{bmatrix} \end{array}$ $\therefore$ W is a subspace of $\mathbb{R}^4$

b) From (a), we see that $W = \{(-r+t, r, s, t) \mid r, s, t \in \mathbb{R}\}$

$\therefore \{\underbrace{(-1,1,0,0)}_{v_1}, \underbrace{(0,0,1,0)}_{v_2}, \underbrace{(1,0,0,1)}_{v_3}\}$ is a basis for W

c) G-S: $w_1 = v_1 = (-1,1,0,0)$.

$$w_2 = v_2 - \frac{v_2 \cdot w_1}{\|w_1\|^2} \cdot w_1 = v_2 = (0,0,1,0)$$

$$w_3 = v_3 - \frac{v_3 \cdot w_1}{\|w_1\|^2} w_1 - \frac{v_3 \cdot w_2}{\|w_2\|^2} \cdot w_2$$

$$= (1,0,0,1) - \frac{(-1)}{2} w_1 - 0\, w_2$$

$$= (1,0,0,1) + \tfrac{1}{2}(-1,1,0,0)$$

$$= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 1).$$ Let's take $\tilde{w}_3 = (1,1,0,2)$

$\therefore \{w_1, w_2, \tilde{w}_3\}$ is an orthog. basis for W

d) $$\mathrm{proj}_W(1,1,1,0) = \frac{\cancel{(1,1,1,0)\cdot(-1,1,0,0)}^{0}}{2}(-1,1,0,0) + \frac{(1,1,1,0)\cdot(0,0,1,0)}{2} w_2$$

$$+ \frac{(1,1,1,0)\cdot(1,1,0,2)}{6} \cdot \tilde{w}_3$$

$$= \tfrac{1}{2} w_2 + \tfrac{1}{3}\tilde{w}_3 = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}\right)$$

13. Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 0 & 4 \end{bmatrix}$.

a) Compute the characteristic polynomial of A and factor it to show that the only eigenvalues of A are 2 and 3.

b) Find a basis of $E_2 = \{v \in \mathbf{R}^3 \mid Av = 2v\}$.

c) Find a basis of $E_3 = \{v \in \mathbf{R}^3 \mid Av = 3v\}$.

d) If possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If A is diagonalizable, explain why your choice of P is invertible. If A is not diagonalizable, explain why.

a) $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 1 & 3-\lambda & 1 \\ 1 & 0 & 4-\lambda \end{vmatrix} \overset{\text{col}_2}{=} (3-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix}$

$= (3-\lambda)\{\lambda^2 - 5\lambda + 4 + 2\}$

$= (3-\lambda)(\lambda-3)(\lambda-2)$

$\therefore$ evals are 2 and 3.

b) $E_2 = \ker(A - 2I) = \ker\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \{(-2s, s, s) \mid s \in \mathbb{R}\}$

$\therefore \{\overset{v_1}{(-2, 1, 1)}\}$ is a basis for E_2.

c) $E_3 = \ker(A - 3I) = \ker\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \{(-t, s, t) \mid s, t \in \mathbb{R}\}$

$\therefore \{\overset{v_2}{(-1, 0, 1)}, \overset{v_3}{(0, 1, 0)}\}$ is a basis for E_3

d) set $P = [v_1\ v_2\ v_3]$ (block col form) $= \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ & $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Then $\det P = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1$ so P is invertible and $P^{-1}AP = D$

(We also know P is invertible because $\dim E_2 + \dim E_3 = 3 = \dim \mathbb{R}^3$)

14. Define a linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$S(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+y \\ y+z \\ z-x \end{bmatrix}.$$

a) Find the standard matrix of S.

b) Find a basis for im S.

c) Give a complete geometric description of im S.

d) Find the dimension of ker S.

a) Let $A = [S(e_1) \ S(e_2) \ S(e_3)] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$. This is
(block col. form)
the standard matrix of S.

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b) Since im S = col A, we find a basis for col A

$A \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so $\{(1,0,-1), (1,1,0)\}$

is a basis for im S

c) From (b), we know S is the plane through 0 in $\mathbb{R}^3$

with normal $(1,0,-1) \times (1,1,0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = (1,-1,1)$.

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d) Since ker A = ker S, dim ker S = dim ker A

$= 3 - \text{rank } A$

$= 3 - 2$

$= 1$.

15 (a). State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you **must give an explicit example - with numbers - to show this.**
- If you say the statement is true, you must give a clear explanation - by quoting a theorem presented in class, or **by giving a proof valid for every case.**

(i) If A is a 2 by 2 matrix and A is diagonalizable, then A is invertible.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, being diagonal, is diagonalizable. $\boxed{\text{False}}$
However, it is not invertible, as rank A is clearly $1 < 2$.

(ii) Suppose B is an invertible 3×3 matrix, and that $\{v_1, v_2, v_3\}$ spans $\mathbf{R}^3$. Then $\{Bv_1, Bv_2, Bv_3\}$ also spans $\mathbf{R}^3$.

Since $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$, if $A = [v_1 \; v_2 \; v_3]$, (block col. form) $\boxed{\text{TRUE.}}$
we know A is invertible.
Then BA is also invertible (as it's the product of 2 invertible matrices). But $BA = B[v_1 \; v_2 \; v_3] = [Bv_1 \; Bv_2 \; Bv_3]$,
So $\{Bv_1, Bv_2, Bv_3\}$ spans $\mathbb{R}^3$ (By the "Invertible matrix theorem")

15 (b). Let A be a real $n \times n$ matrix. Give 3 statements (in total) equivalent to

"A is **not** invertible",

one each in terms of:

(I) the columns of A

- are linearly dependent
- do not span $\mathbb{R}^n$
- are not a basis of $\mathbb{R}^n$

(only one is necessary)

(II) the determinant of A

is zero.

(III) the homogeneous linear system $Ax = 0$, where $x \in \mathbf{R}^n$ will have

infinitely many solutions

16. (4 bonus marks) Make sure you finish and check the rest of the paper before trying this. Bonus marks are much harder to earn.

Suppose $\{u, v, w\} \subset \mathbf{R}^3$ is a set of non-zero vectors.

Prove the following carefully. **N.B. Your proof must be valid for <u>every</u> set $\{u, v, w\}$ of non-zero vectors in $\mathbf{R}^3$, so don't choose any of them yourself!**

If $|u \cdot v \times w| = \|u\|\,\|v\|\,\|w\|$, then $\{u, v, w\}$ is an orthogonal set.

Let θ be the angle between u & $v \times w$ and φ be the angle between v and w.

Then $|u \cdot v \times w| = \|u\|\,\|v \times w\|\,|\cos\theta|$

$= \|u\|\,\|v\|\,\|w\| \sin\varphi\, |\cos\theta|$

$= \|u\|\,\|v\|\,\|w\|.$

Since $0 \le |\cos\theta|, \sin\varphi \le 1$, this implies $\sin\varphi = 1$ and $|\cos\theta| = 1$. Thus $\varphi = \pi/2$, so v and w are orthogonal.

Now use the fact that $|u \cdot v \times w| = |u \times v \cdot w| = |w \times u \cdot v|$ and argue as above to conclude that u and v and u and w are also orthogonal.

[Or: Since $|\cos\theta| = 1$, u is parallel to $v \times w$ and so is perpendicular to both!!