ENCS
Engineering & Computer Science

ENGR 213
Applied Ordinary Differential Equations
First Mid Term Exam
February 5th 2017

Student Name: Andrew Wong

Student ID Number: 40044916

ENGR 213 Section: 4-S15 G

Course Given BY: Iman Gohar

Q1	8
Q2	10
Q3	4
Q4	10
Q5	10
Total	42

- **Exam is closed book close notes.**
- **No use of any electronic devices.**
- **Use only the approved calculator.**
- **Write your answers in the provided space.**

Problem#1

Classify the following differential equation by order and linearity.

a) $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^4$

b) $y'' = \sqrt{5x^6 + (\sin x)y^{(8)}}$

c) $e^{-x}y' + (4\sin x)y = (\cancel{\tan x})y'' - 4e^{-x^2}$

d) $\frac{dy}{dt} = 1 - t^2$

e) $x^2 y'' + x y' + (x^2 - v^2)y = 0$

Solution:

	Order	**Linearity**
a)	second order	non linear
b)	second order	non linear
c)	second order	non linear
d)	first order	linear
e)	second order	linear

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Problem#2:

Show that $y = \frac{1}{4}e^{3x} + ce^{-x}$ is a solution to the DE $\frac{dy}{dx} + y = e^{3x}$. What is the largest interval of existence for a solution to the IVP *y (-3)=15?*

Solution:

$y' = \frac{3}{4}e^{3x} - ce^{-x} \rightarrow \frac{3}{4}e^{3x} - ce^{-x} + y = e^{3x}$

$\rightarrow \frac{3}{4}e^{3x} \cancel{+ce^{-x}} + \frac{1}{4}e^{3x} \cancel{+ce^{-x}} = e^{3x}$ $\qquad \frac{4}{4}e^{3x} = e^{3x}$

yes y is a solution to the DE

$y = \frac{1}{4}e^{3x} + \frac{c}{e^{x}}$ $\qquad 15 = \frac{1}{4}e^{-9} + ce^{3}$

$15 - \frac{1}{4}e^{-9} = ce^{3}$ $\qquad c = \frac{15 - \frac{1}{4}e^{-9}}{e^{3}} = 0.7468..$

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$y = \frac{1}{4}e^{3x} + \boxed{\frac{0.7468}{e^{x}}} \rightarrow e^{-x}$ has domain $(-\infty, \infty)$

solution exists for all values of x

largest interval is $(-\infty, \infty)$

$y^2-3y-1y+3$

Problem#3

$y(y-3)-1(y+3)$

Solve the differential equation $y' = y^2 - 4y + 3$ given initial condition $y(0) = 3$.

Solution:

$$\frac{dy}{dx} = y^2 - 4y + 3 \qquad \frac{dy}{dx} = (y-2)^2 + 7 = y^2 - 4y + 4 + 7$$

?

$$\int \frac{1}{(y-2)^2+7}\,dy = \int dx \qquad (2) = \int dx = x + C$$

(1) (2)

Wrong

Does not work!

$$(1)\ \int \frac{1}{(y-2)^2+7}\,dy \qquad u = y-2,\ du = dx \qquad \int \frac{1}{u^2+7}\,du = \int \frac{1}{7\left(\frac{u^2}{7}+1\right)}\,du$$

$$\frac{1}{7}\int \frac{du}{\left(\frac{u}{\sqrt{7}}\right)^2+1} \qquad \frac{1}{7}\arctan\left(\frac{u}{\sqrt{7}}\right) = \frac{1}{7}\arctan\left(\frac{y-2}{\sqrt{7}}\right)$$

$$\frac{1}{7}\arctan\left(\frac{y-2}{\sqrt{7}}\right) = x + C$$

$$\frac{1}{7}\arctan\left(\frac{3-2}{\sqrt{7}}\right) = 0 + C$$

$$C = \frac{1}{7}\arctan\left(\frac{1}{\sqrt{7}}\right) \overset{\text{in rad}}{=} 0.05162$$

$$\frac{1}{7}\arctan\left(\frac{y-2}{\sqrt{7}}\right) - x = 0.05162$$

2

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Solution

$$\frac{dy}{dx} = (y-2)^2 - 1$$

$$\int \frac{dy}{(y-2)^2-1} = \int dx$$

$$[(y-2)-1]((y-2)+1)$$

$$(y-3)(y-1)$$

$$\frac{1}{(y-3)(y-1)} = \frac{A}{y-3} + \frac{B}{y-1}$$

$$1 = A(y-1) + B(y-3)$$

$$y = 1 \qquad 1 = -2B$$

Problem#4

Solve the initial value problem $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}$, $y(0) = 2$

Solution

$$y(1-x^2)\,dy = (xy^2 - \cos x \sin x)\,dx$$

$$y(1-x^2)\,dy + (-xy^2 + \cos x \sin x)\,dx = 0$$

$$(y - yx^2)\,dy + (-xy^2 + \cos x \sin x)\,dx = 0$$

$$N_x = -2xy = M_y = -2xy$$

$$f(x,y) = \int N(y)\,dy = \int y - yx^2\,dy = \frac{1}{2}y^2 - \frac{1}{2}y^2x^2 + h(x)$$

$$\frac{\partial f}{\partial x} = -y^2x + h'(x) = M = -xy^2 + \cos x \sin x$$

$$h'(x) = \cos x \sin x$$

$$h(x) = \int \cos x \sin x\,dx = \quad u = \sin x,\ du = \cos x\,dx \quad \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}\sin^2 x$$

$$f(x,y) = \frac{1}{2}y^2 - \frac{1}{2}y^2x^2 + \frac{1}{2}\sin^2 x = c$$

$$\frac{1}{2}(4) - \frac{1}{2}(4)(0) + \frac{1}{2}(0) = c$$

$$c = 2$$

$$f(x,y) = \frac{1}{2}y^2 - \frac{1}{2}y^2x^2 + \frac{1}{2}\sin^2 x = 2$$

Problem#5

Solve the differential equation $\frac{dx}{dy} = x(yx^3 - 1)$

Solution:

$\frac{dx}{dy} = yx^4 - x \qquad \frac{dx}{dy} + x = yx^4 \qquad \frac{1}{x^4}\frac{dx}{dy} + \frac{1}{x^3} = y$

$v = \frac{1}{x^3} \qquad v' = -\frac{3}{x^4}x' \qquad -\frac{1}{3}\frac{dv}{dy} + v = y \qquad \frac{dv}{dy} - 3v = -3y$

$p(b) = -3 \qquad \int -3\,dy = -3y \rightarrow e^{-3y}$

$e^{-3y}\frac{dv}{dy} - 3e^{-3y}v = -3y e^{-3y}$

$\int (e^{-3y}v)'\,dv = \int -3y e^{-3y}\,dy$ (1)

(1) $u = -3y$, $du = -3\,dy$ $\quad -\frac{1}{3}\int u e^{u}\,du$

	du	∫
+	u	e^u
−	1	e^u
+	0	e^u

$-\frac{1}{3}\left[ue^u - e^u\right] = -\frac{1}{3}\left(-3y e^{-3y} - e^{-3y} + c\right)$

$e^{-3y}v = ye^{-3y} + \frac{e^{-3y}}{3} + c_1$

$v = y + \frac{1}{3} + c_1 e^{3y}$

$\frac{1}{x^3} = y + \frac{1}{3} + c_1 e^{3y}$

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Concordia University
ENCS
Engineering & Computer Science

ENGR 213

Applied Ordinary Differential Equations

Second Mid Term Exam

March 12th 2017

Student Name: Andrew Wong

Student ID Number: 40044916

ENGR 213 Section: G

Course Given BY: Iman Gohar.

Q1	10
Q2	10
Q3	10
Q4	10
Q5	8
Total	48

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Problem#1

Write the number $(3+6i)+(4\text{-}i)(3+5i)+\frac{1}{2-i}$ in the form $a+bi$.

1 2 3

Solution:

2 $(4-i)(3+5i)$

$12+20i-3i-5i^2$

$12+17i+5 = 17+17i$ ✓

3 $\frac{1}{2-i} \cdot \frac{2+i}{2+i} = \frac{2+i}{4-i^2} = \frac{2+i}{5} = \frac{2}{5}+\frac{i}{5}$ ✓

$3+6i+17+17i+\frac{2}{5}+\frac{i}{5} = \frac{102}{5}+\frac{116}{5}i$ ✓

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Problem#2

Verify that $y_1 = e^{5x}$ and $y_2 = e^{-7x}$ are form a fundamental set of solutions of the differential equation $y''+2y'-35y=0$ on the interval $(-\infty,\infty)$. Write the form of the general solution.

Solution:

$y_1 = e^{5x}$ | $y_2 = e^{-7x}$

$y_1' = 5e^{5x}$ | $y_2' = -7e^{-7x}$

$y_1'' = 25e^{5x}$ | $y_2'' = 49e^{-7x}$

$$W = \begin{vmatrix} e^{5x} & e^{-7x} \\ 5e^{5x} & -7e^{-7x} \end{vmatrix} = -7e^{-2x} + 5e^{-2x} = -2e^{-2x}$$

$\neq 0$

$25e^{5x} + 10e^{5x} - 35e^{5x} = 0$

$35e^{5x} - \cancel{35e^{5x}} = 0$

$49e^{-7x} - 14e^{-7x} - 35e^{-7x} = 0$

$49e^{-7x} - 49e^{-7x} = 0$

$y_g = c_1 e^{5x} + c_2 e^{-7x}$

$y_g' = 5c_1 e^{5x} - 7c_2 e^{-7x}$

$y_g'' = 25c_1 e^{5x} + 49c_2 e^{-7x}$

$25c_1 e^{5x} + 49c_2 e^{-7x} + 10c_1 e^{5x} - 14c_2 e^{-7x} - 35c_1 e^{5x} - 35c_2 e^{-7x} = 0$

$c_1 = 1$

$c_2 = 1$

$y_g = c_1 e^{5x} + c_2 e^{-7x}$

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Problem#3

Solution:

(a) Write the characteristic equation and the general solution of the differential equation:

$y^{(6)} - 13y^{(5)} + 70y^{(4)} - 198y^{(3)} + 308y'' - 248y' + 80y = 0.$

$y^{(k)}$ is the k-th derivative of y. For Your convenience, the roots of the characteristic equation are: 1, 2, 2, 2, 3 + i, 3 – i. You do not have to check this.

(b) Find the general solution of the differential equation: $x^2 y'' - 7xy' + 16y = 0$, given one solution $y_1 = x^4$.

a) **Solution:**

$y_1 = e^{x}$ $y_2 = e^{2x}$ $y_3 = xe^{2x}$ $y_4 = x^2 e^{2x}$ $y_5 = e^{3x}(\cos x + i\sin x)$ $y_6 = e^{3x}(\cos x - i\sin$

$y_g = C_1 e^{x} + C_2 e^{2x} + C_3 x e^{2x} + C_4 x^2 e^{2x} + e^{3x}(C_5 \cos x + C_6 \sin x)$ 3

b) $x^2 y'' - 7xy' + 16y = 0$

$m(m-1) - 7m + 16 = 0$

$m^2 - m - 7m + 16 = 0$

$m^2 - 8m + 16 = 0$

$(m-4)^2 = 0$

$m_1 = 4 = m_2$

$y_1 = x^4$

$y_2 = x^4 \ln x$

$y_g = C_1 x^4 + C_2 x^4 \ln x$

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Problem#4

Using the method of undetermined coefficients, solve the initial value problem

y''+y'-2y= (6x+2) e^x; y(0)=0; y'(0)=0.

Solution:

solve homogeneous ;

$m^2 + m - 2 = 0$

$(m-1)(m+2) = 0$ $\quad y_1 = e^x$ $\quad y_c = c_1 e^x + c_2 e^{-2x}$

$m_1 = 1 \quad m_2 = -2$ $\quad y_2 = e^{-2x}$

$y_p = (Ax+B)e^x$ → duplicate $(Ax^2+Bx)e^x$

$= (Ax^2+Bx)e^x$

$y_p' = (2Ax+B)e^x + (Ax^2+Bx)e^x = e^x(2Ax+B+Ax^2+Bx)$

$y_p'' = (2A)e^x + (2Ax+B)e^x + (2Ax+B)e^x + (Ax^2+Bx)e^x$

$= e^x(2A + 4Ax + 2B + Ax^2 + Bx)$

$\cancel{e^x}(2A+4Ax+2B+Ax^2+Bx) + \cancel{e^x}(2Ax+B+Ax^2+Bx) - \cancel{e^x}(2Ax^2+2Bx) = (6x+2)\cancel{e^x}$

$2A+4Ax+2B+Ax^2+\cancel{Bx}+2Ax+B+Ax^2+\cancel{Bx}-2Ax^2-2\cancel{Bx} = 6x+2$

$2A + 6Ax + 3B = 6x+2$ $\quad | \quad 6Ax = 6x$ $\quad \boxed{A=1}$

$2 + 6x + 3B = 6x+2$ $\quad 3B = 0$ $\quad \boxed{B=0}$

$y_p = x^2e^x$ $\quad y_g = c_1e^x + c_2e^{-2x} + x^2e^x$ $\quad y_g' = c_1e^x - 2c_2e^{-2x} + 2xe^x + x^2e^x$

$y(0)=0$ $\quad 0 = c_1e^0 + c_2e^0 + 0$ $\quad | \; y'(0)=0$ $\quad 0 = c_1e^0 - 2c_2e^0 + 0 + 0$

$0 = c_1 + c_2$ $\quad 0 = c_1 - 2c_2$

$c_1 = 2c_2$

$0 = 2c_2 + c_2$

$0 = 3c_2$

$c_2 = 0$

$c_1 = 0$

$y_g = x^2e^x$

Problem#5

Solve the following differential Equation $y'' + (3/x)\, y' + (5/x^2)\, y = 0$

Solution:

$\left(y'' + \frac{3}{x} y' + \frac{5}{x^2} y = 0\right) x^2 \qquad x^2 y'' + 3xy' + 5y = 0$

$m(m-1) + 3m + 5 = 0$

$m^2 - m + 3m + 5 = 0$

$m^2 + 2m + 5 = 0$ ✓ 5

✓ 3

$\frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{-1 \pm \sqrt{4 - 20}}{2} = -1 \pm \frac{\sqrt{-16}}{2} = -1 \pm 2i$

$y_1 = e^{-x}(\cos 2x + i \sin 2x) \qquad y_2 = e^{-x}(\cos 2x - i \sin 2x)$

$y_g = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$ Form Wrong!

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