MAT 2384 C
DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS
MIDTERM

February 12, 2014

Duration: 80 minutes

Name: $\boxed{\text{Solutions}}$

Student Number:

Instructions:

- Print your name and student number on this page.
- Verify that your copy of the exam has all 5 pages.
- You must answer all questions.
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary.
- **No Notes or Books.**
- **Basic scientific calculators only - graphing and/or programmable calculators are NOT permitted.**

Question 1 (6 marks) Find the general solution of

$$(4xy - e^x y^2)\, dx + (4x^2 - 3e^x y)\, dy = 0\,.$$

$$\left.\begin{array}{ll} M(x,y) = 4xy - e^x y^2 & \Rightarrow\ M_y = 4x - 2e^x y \\ N(x,y) = 4x^2 - 3e^x y & \Rightarrow\ N_x = 8x - 3e^x y \end{array}\right\} \begin{array}{l} M_y \neq N_x \\ \text{DE \underline{not} exact} \end{array}$$

$$\frac{M_y - N_x}{M} = \frac{-4x + e^x y}{4xy - e^x y^2} = -\frac{1}{y}\,, \text{ (a function of } y \text{ only)}$$

then $\mu(y) = e^{-\int -\frac{1}{y}\,dy} = e^{\int \frac{1}{y}\,dy} = e^{\ln y} = y$ and the DE becomes $(4xy^2 - e^x y^3)\,dx + (4x^2 y - 3e^x y^2)\,dy = 0$

$$\left.\begin{array}{ll} M^*(x,y) = 4xy^2 - e^x y^3 & \Rightarrow\ M^*_y = 8xy - 3e^x y^2 \\ N^*(x,y) = 4x^2 y - 3e^x y^2 & \Rightarrow\ N^*_x = 8xy - 3e^x y^2 \end{array}\right\} \begin{array}{l} M^*_y = N^*_x \\ \text{DE is exact} \end{array}$$

$$F(x,y) = \int M^*(x,y)\,dx + g(y)$$
$$= \int (4xy^2 - e^x y^3)\,dx + g(y) = 2x^2y^2 - e^x y^3 + g(y)$$

then $\dfrac{\partial F}{\partial y} = 4x^2 y - 3e^x y^2 + g'(y) = N^*(x,y) = 4x^2 y - 3e^x y^2$

thus $g'(y) = 0 \Rightarrow$ take $g(y) = 0$

and so $F(x,y) = 2x^2y^2 - e^x y^3$

$\therefore$ the general solution is $\boxed{2x^2y^2 - e^x y^3 = C}$

Question 2 (10 marks) Solve the initial value problems.

(a) $\dfrac{dy}{dx} = \dfrac{e^{-y}}{\sqrt{1-x^2}}$, $y(0) = 0$

DE is separable : $e^y dy = \dfrac{dx}{\sqrt{1-x^2}}$

integrate on both sides $\int e^y dy = \int \dfrac{dx}{\sqrt{1-x^2}} + C$

to get $e^y = \arcsin x + C$

so the general solution is $y = \ln|\arcsin x + C|$

$y(0) = 0 \Rightarrow 0 = \ln(\arcsin(0) + C) = \ln C \Rightarrow C = 1$

$\therefore$ the unique solution is $\boxed{y = \ln|1 + \arcsin x|}$

(b) $y' + y = y^2$, $y(0) = 1/2$

This is Bernoulli with $p(x) = g(x) = 1$ and $a = 2$

so let $u = y^{-1}$ and the DE becomes $u' - u = -1$

then $\mu(x) = e^{-\int dx} = e^{-x}$

so $u = e^x\left(\int -e^{-x}dx + C\right) = e^x(e^{-x} + C) = 1 + Ce^x$

and so the general solution is $y = (1 + Ce^x)^{-1} = \dfrac{1}{1+Ce^x}$

$y(0) = 1/2 \Rightarrow 1/2 = \dfrac{1}{1+C} \Rightarrow C = 1$

$\therefore$ the unique solution is $\boxed{y = \dfrac{1}{1+e^x}}$

(c) $y'' - 7y' + 12y = 0$, $y(0) = 7$, $y'(0) = 25$

The characteristic equation is $\lambda^2 - 7\lambda + 12 = (\lambda-3)(\lambda-4) = 0$

so the roots are $\lambda_1 = 3$, $\lambda_2 = 4$ and the general solution is $y(x) = C_1 e^{3x} + C_2 e^{4x}$

$$\left.\begin{array}{l} y(0) = 7 \Rightarrow 7 = C_1 e^0 + C_2 e^0 \Rightarrow C_1 + C_2 = 7 \\ y'(x) = 3C_1 e^{3x} + 4C_2 e^{4x} \\ y'(0) = 25 \Rightarrow 25 = 3C_1 e^0 + 4C_2 e^0 \Rightarrow 3C_1 + 4C_2 = 25 \end{array}\right\} \begin{array}{l} C_2 = 4 \\ C_1 = 3 \end{array}$$

$\therefore$ the unique solution is $\boxed{y(x) = 3e^{3x} + 4e^{4x}}$

A

Question 3 (7 marks) Find the general solutions of the differential equations.
(a) $y'' + 2\pi y' + \pi^2 y = 0$

the char. eq. is $\lambda^2 + 2\pi\lambda + \pi^2 = (\lambda+\pi)^2 = 0$

so the roots are $\lambda_1 = \lambda_2 = -\pi$

$\therefore$ the general solution is $\boxed{y(x) = C_1 e^{-\pi x} + C_2 x e^{-\pi x}}$

(b) $x^2 y'' - xy' + 10y = 0, \quad x > 0$

the char. eq. is $m(m-1) - m + 10 = 0$

or $m^2 - 2m + 10 = 0$

the roots are $m_{1,2} = \dfrac{2 \pm \sqrt{(-2)^2 - 4(10)}}{2} = \dfrac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$

$\therefore$ the general solution is $\boxed{y(x) = C_1 x \cos(3\ln x) + C_2 x \sin(3\ln x)}$

(c) $y''' + 4y'' + y' - 6y = 0$

the char. eq. is $\lambda^3 + 4\lambda^2 + \lambda - 6 = 0$

by inspection $\lambda = 1$ is a root, so

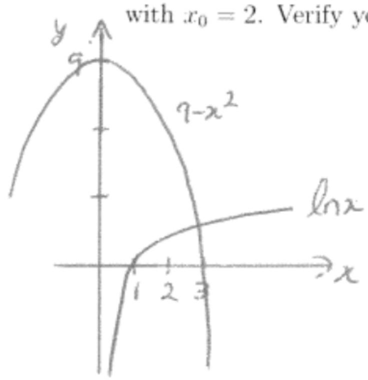
$\lambda^3 + 4\lambda^2 + \lambda - 6 = (\lambda-1)(\lambda^2 + 5\lambda + 6) = (\lambda-1)(\lambda+2)(\lambda+3)$

and the roots are $\lambda_1 = 1, \ \lambda_2 = -2, \ \lambda_3 = -3$

$\therefore$ the general solution is $\boxed{y(x) = C_1 e^{x} + C_2 e^{-2x} + C_3 e^{-3x}}$

A

Question 4 (7 marks)
(a) Use Newton's Method to find the solution of $\ln x = 9 - x^2$ to four decimal places. Start with $x_0 = 2$. Verify your answer.



let $f(x) = \ln x - (9 - x^2) = \ln x - 9 + x^2$

then $f'(x) = \frac{1}{x} + 2x$

Newton's Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

so $x_{n+1} = x_n - \dfrac{\ln x_n - 9 + x_n^2}{\frac{1}{x_n} + 2x_n} = \dfrac{10 - \ln x_n + x_n^2}{\frac{1}{x_n} + 2x_n}$

$x_0 = 2, \quad x_1 = \dfrac{10 - \ln(2) + (2)^2}{\frac{1}{2} + 2(2)} = 2.9571$

$x_2 = 2.8246$

$x_3 = 2.8218 = x_4 \quad \therefore$ stop

check: $f(2.8218) \simeq -7 \times 10^{-5}$ okay

$\therefore$ solution is $\boxed{2.8218}$

(b) If we were going to find the interpolating polynomial $p_n(x)$ for 5 data points, what is the maximum possible degree that the polynomial could have? Circle the letter of the correct answer.

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6 (F) we cannot tell

$\begin{pmatrix} n+1=5 \\ n=4 \end{pmatrix}$

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Question 1 (6 marks) Find the general solution of

$$(3x^2y^2 + 2e^x y)\,dx + (3x^3y + 4e^x)\,dy = 0\,.$$

$M(x,y) = 3x^2y^2 + 2e^x y \Rightarrow M_y = 6x^2y + 2e^x$

$N(x,y) = 3x^3y + 4e^x \Rightarrow N_x = 9x^2y + 4e^x$

$\Big\}$ $M_y \neq N_x$ DE ~~not~~ exact

$$\frac{M_y - N_x}{M} = \frac{-3x^2y - 2e^x}{3x^2y^2 + 2e^x y} = \frac{-1}{y} \Rightarrow \mu(y) = y \text{ and DE}$$

becomes $(3x^2y^3 + 2e^x y^2)dx + (3x^3y^2 + 4e^x y)dy = 0$

$M^*(x,y) = 3x^2y^3 + 2e^x y^2 \Rightarrow M^*_y = 9x^2y^2 + 4e^x y$

$N^*(x,y) = 3x^3y^2 + 4e^x y \Rightarrow N^*_x = 9x^2y^2 + 4e^x y$

$\Big\}$ $M^*_y = N^*_x$ DE exact

$$F(x,y) = \int N^*(x,y)\,dy + g(x)$$

$$= \int (3x^3y^2 + 4e^x y)\,dy + g(x) = x^3y^3 + 2e^x y^2 + g(x)$$

then $\dfrac{\partial F}{\partial x} = 3x^2y^3 + 2e^x y^2 + g'(x) = M^*(x,y) = 3x^2y^3 + 2e^x y^2$

So $g'(x) = 0 \Rightarrow$ take $g(x) = 0$

thus $F(x,y) = x^3y^3 + 2e^x y^2$

$\therefore$ the general solution is $\boxed{x^3y^3 + 2e^x y^2 = C}$

Question 2 (10 marks) Solve the initial value problems.

(a) $\dfrac{dy}{dx} = \dfrac{e^{-y}}{1+x^2}$, $y(0) = 1$

DE is separable $e^y\,dy = \dfrac{dx}{1+x^2}$

so $\int e^y dy = \int \dfrac{dx}{1+x^2} + C$

we get $e^y = \arctan x + C$

the general solution is $y = \ln|\arctan x + C|$

$y(0) = 1 \Rightarrow 1 = \ln|\arctan(0) + C| = \ln C \Rightarrow C = e$

$\therefore$ the unique solution is $\boxed{y = \ln|e + \arctan x|}$

(b) $y' + y = y^2$, $y(0) = 1/3$

(see version A)

the general solution is $y = \dfrac{1}{1+Ce^x}$

then $y(0) = 1/3 \Rightarrow \dfrac{1}{3} = \dfrac{1}{1+C} \Rightarrow C = 2$

$\therefore$ the unique solution is $\boxed{y = \dfrac{1}{1+2e^x}}$

(c) $y'' - 9y' + 20y = 0$, $y(0) = 9$, $y'(0) = 41$

the char. eq. is $\lambda^2 - 9\lambda + 20 = (\lambda-4)(\lambda-5) = 0$

so the roots are $\lambda_1 = 4$, $\lambda_2 = 5$ and the general solution is $y(x) = C_1e^{4x} + C_2e^{5x}$

$y(0) = 9 \Rightarrow 9 = C_1e^0 + C_2e^0 \Rightarrow C_1 + C_2 = 9$

$y'(x) = 4C_1e^{4x} + 5C_2e^{5x}$

$y'(0) = 41 \Rightarrow 41 = 4C_1e^0 + 5C_2e^0 \Rightarrow 4C_1 + 5C_2 = 41$

$\Big\}\ C_2 = 5,\ C_1 = 4$

$\therefore$ the unique solution is $\boxed{y(x) = 4e^{4x} + 5e^{5x}}$

B

Question 3 (7 marks) Find the general solutions of the differential equations.

(a) $y'' - 2\pi y' + \pi^2 y = 0$

the char. eq. is $\lambda^2 - 2\pi\lambda + \pi^2 = (\lambda - \pi)^2 = 0$

so the roots are $\lambda_1 = \lambda_2 = \pi$

$\therefore$ the general solution is $\boxed{y(x) = C_1 e^{\pi x} + C_2 x e^{\pi x}}$

(b) $x^2 y'' - 3xy' + 13y = 0, \quad x > 0$

the char. eq. is $m(m-1) - 3m + 13 = 0$

or $m^2 - 4m + 13 = 0$

the roots are $m_{1,2} = \dfrac{4 \pm \sqrt{(-4)^2 - 4(13)}}{2} = \dfrac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$

$\therefore$ the general solution is $\boxed{y(x) = C_1 x^2 \cos(3\ln x) + C_2 x^2 \sin(3\ln x)}$

(c) $y''' + 2y'' - 5y' - 6y = 0$

the char. eq. is $\lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0$

by inspection, $\lambda = -1$ is a root, so

$\lambda^3 + 2\lambda^2 - 5\lambda - 6 = (\lambda + 1)(\lambda^2 + \lambda - 6) = (\lambda + 1)(\lambda + 3)(\lambda - 2)$

and the roots are $\lambda_1 = -1$, $\lambda_2 = -3$ and $\lambda_3 = 2$

$\therefore$ the general solution is $\boxed{y(x) = C_1 e^{-x} + C_2 e^{-3x} + C_3 e^{2x}}$

B

Question 4 (7 marks)
(a) Use Newton's Method to find the solution of $\ln x = 8 - x^2$ to four decimal places. Start with $x_0 = 2$. Verify your answer.

$$f(x) = \ln x - 8 + x^2, \quad f'(x) = \frac{1}{x} + 2x$$

$$x_{n+1} = x_n - \frac{\ln x_n - 8 + x_n^2}{\frac{1}{x_n} + 2x_n} = \frac{9 - \ln x_n + x_n^2}{\frac{1}{x_n} + 2x_n}$$

$$x_0 = 2, \quad x_1 = \frac{9 - \ln 2 + (2)^2}{\frac{1}{2} + 2(2)} = 2.7349$$

$$x_2 = 2.6517$$

$$x_3 = 2.6505 = x_4 \quad \therefore \text{stop}$$

Check: $f(2.6505) \simeq -1 \times 10^{-4}$ okay

$\therefore$ solution is $\boxed{2.6505}$

(b) If we were going to find the interpolating polynomial $p_n(x)$ for 4 data points, what is the maximum possible degree that the polynomial could have? Circle the letter of the correct answer.

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6 (F) we cannot tell

$\begin{pmatrix} n+1=4 \\ n=3 \end{pmatrix}$